A New Proof of the Caffarelli-Kohn-Nirenberg Theorem

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Abstract

Here we give a self-contained new proof of the partial regularity theorems for solutions of incompressible Navier-Stokes equations in three spatial dimensions. These results were originally due to Scheffer and Caffarelli, Kohn, and Nirenberg. Our proof is much more direct and simpler. © 1998 John Wiley & Sons, Inc.

1 Introduction

This paper is concerned with the partial regularity of weak solutions of the incompressible Navier-Stokes equations in three spatial dimensions with unit viscosity and zero external force:

$$\begin{cases} v_t + v \cdot \nabla v - \Delta v + \nabla p = 0\,, & v(x,t) \in \mathbb{R}^3\,, \\ \operatorname{div} v = 0\,. \end{cases} \tag{1.1}$$

Of particular interest is the initial boundary value problem on a bounded, smooth domain $\Omega \subseteq \mathbb{R}^3$. In addition to (1.1) on $\Omega \times (0,t)$, one requires

$$\begin{cases} v(x,0) = v_0(x)\,, & x \in \Omega\,, \\ v(x,t) = 0\,, & x \in \partial\Omega\,,\ 0 < t < T\,. \end{cases} \tag{1.2}$$

Here the initial data should satisfy

$$v_0(x) = 0\,, \quad x \in \partial\Omega\,, \quad \text{and} \quad \operatorname{div} v_0 = 0 \text{ in } \Omega\,. \tag{1.3}$$

The concepts of weak solutions of (1.1)–(1.2) and their regularity were already introduced in the fundamental paper of J. Leray [10]. Pioneering works of Leray [10] and Hopf [5] showed the existence of a function v and a distribution P such that

(i) $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ for each $T < \infty$;

(ii) equations (1.1) hold weakly;

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(iii)

$$\int_{\Omega\times\{t\}} |v|^2\,dx + 2\int_0^t\int_\Omega |\nabla v|^2\,dx\,dt \le \int_\Omega |v_0|^2\,dx$$

and

$$\lim_{t\to 0^+} \|v(\,\cdot\,,t) - v_0(\,\cdot\,)\| = 0$$

whenever $v_0 \in L^2(\Omega)$ (cf. [22, chap. III]).

There are many important results concerning the regularity of weak solutions. Among them we wish to mention the work of Serrin [19], which asserts that if a weak solution v of (1.1) also satisfies that $v \in L^p(0,T;L^q(\Omega))$ for some $p,q \ge 1$ so that $2/p + 3/q < 1$, then v is smooth in the spatial direction. This result was later improved in [21] and [2] to the case of equality.

It is also well-known that if v_0 is smooth enough, then problems (1.1)–(1.3) have a unique solution on $\Omega \times (0,T)$ for some $T > 0$; see, for example, [7], [4], [6], [13], [23], and the references therein.

In a series of papers [15, 16, 17, 18], Scheffer introduced the notions of "suitable weak solutions" and the "generalized energy inequality." He established various partial regularity results for such weak solutions; see also an interesting related work of Foias and Temam [3]. Scheffer's results were further generalized and strengthened in the paper of Caffarelli, Kohn, and Nirenberg [1], where the best partial regularity theorem to date was proved.

The purposes of this paper are to further elucidate the main ideas of the partial regularity theory, to discuss a few unsolved issues in [1], and to give simpler proofs of the main results of [1]. Part of the simplification is due to the better estimate of Sohr and von Wahl [20] on the pressure P, which was developed after the publication of [1]; see Lemma 2.3 below. Another part of the simplification is due to the fact that we worked with somewhat more natural quantities.

Throughout the paper, we shall use the notation in [22].

2 Preliminaries

There are two key ingredients in the proof of the partial regularity theorem. The first one is the often-used interpolation inequality, which follows:

For $v \in H^1(B_r)$, one has

$$\int_{B_r} |v|^q \, dx \le c \left(\int_{B_r} |\nabla v|^2 \, dx \right)^{q/2-a} \cdot \left(\int_{B_r} |v|^2 \, dx \right)^a + \frac{c}{r^{2a}} \left(\int_{B_r} |v|^2 \, dx \right)^{q/2} \tag{2.1}$$

for all $2 \le q \le 6, \ a = 3(q-2)/4$. Here B_r is a ball of radius r in $\mathbb{R}^3$.

In fact, we also need the following variation of (2.1) (cf. lemma 5.2 in [1]):

LEMMA 2.1 *If* $0 < r \le \rho$, *then*

$$C(r) \le c \left[\left(\frac{r}{\rho} \right)^3 A^{3/2}(\rho) + \left(\frac{\rho}{r} \right)^3 A(\rho)^{3/4} B(\rho)^{3/4} \right]$$

where

$$\begin{aligned} A(r) &= \sup_{-r^2 \le t \le 0} \frac{1}{r} \int_{B_r(0) \times \{t\}} |v|^2 \, dx \,, \\ B(r) &= \frac{1}{r} \int_{Q_r} \int |\nabla v|^2 \, dx \, dt \,, \\ C(r) &= \frac{1}{r^2} \int_{Q_r} \int |v|^3 \, dx \, dt \,, \\ Q_r &= \left\{ (x,t) \in \mathbb{R}^3 \times \mathbb{R} : |x| \le r, \ -r^2 \le t \le 0 \right\}. \end{aligned}$$

PROOF: At almost every time we estimate

$$\begin{aligned} \int_{B_r} |v|^2 \, dx &= \int_{B_r} \left(|v|^2 - |\bar{v}|_\rho^2 \right) dx + \int_{B_r} |\bar{v}|_\rho^2 \, dx \\ &\le \int_{B_\rho} \left| |v|^2 - |\bar{v}|_\rho^2 \right| dx + \int_{B_r} |\bar{v}|_\rho^2 \, dx \\ &\le c\rho \int_{B_\rho} |v| \, |\nabla v| \, dx + c \left(\frac{r}{\rho} \right) \int_{B_\rho} |v|^2 \, dx \end{aligned}$$

using Poincaré's inequality on the first term. Here

$$\bar{f}_\rho = \frac{1}{|B_\rho|} \int_{B_\rho} f \, dx \,.$$

Thus

$$\int_{B_r} |v|^2 \, dx \le c\rho^{3/2} A^{1/2}(\rho) \left(\int_{B_\rho} |\nabla v|^2 \, dx \right)^{1/2} + c \left(\frac{r}{\rho} \right)^3 \rho A(\rho) \,. \tag{2.2}$$

By combining (2.1) and (2.2), one obtains that

$$\int_{B_r} |v|^3\,dx \le c\left(\int_{B_r} |\nabla v|^2\,dx\right)^{3/4}\left(\int_{B_\rho} |v|^2\,dx\right)^{3/4} + \frac{c}{r^{3/2}}\left(\int_{B_\rho} |v|^2\,dx\right)^{3/2} + c\left(\frac{r}{\rho}\right)^{3/4} A^{3/2}(\rho) + c\left(\rho^{3/4} + \frac{\rho^{9/4}}{r^{3/2}}\right) A^{3/4}(\rho)\left(\int_{B_\rho} |\nabla v|^2\,dx\,dt\right)^{3/4}.$$

Therefore, by integrating from $-r^2$ to 0 and applying Hölder's inequality, we obtain that

$$\int_{Q_r}\int |v|^3\,dx\,dt \le cr^2\left(\frac{r}{\rho}\right)^3 A(\rho)^{3/2} + c\left(\rho^{3/4} + \frac{\rho^{9/4}}{r^{3/2}}\right) r^{1/2} A(\rho)^{3/4}\left(\int_{Q_\rho}\int |\nabla v|^2\,dx\,dt\right)^{3/4}.$$

The conclusion of Lemma 2.1 follows. ∎

The second key ingredient is the so-called generalized energy inequality, which makes the Leray-Hopf theory localizable. A weak solution (v, P) of (1.1) is said to satisfy the generalized energy inequality if

$$2\int_0^T\int_\Omega |\nabla v|^2\,\phi\,dx\,dt \le \int_0^T\int_\Omega |v|^2(\phi_t + \Delta\phi)\,dx\,dt + \int_0^t\int_\Omega (|v|^2 + 2P)\,v\cdot\nabla\phi\,dx\,dt \tag{2.3}$$

for all nonnegative $\phi \in C_0^\infty(\Omega \times (0, T))$.

Let us recall that a well-known property of weak solutions is the weak continuity of v as a function of time (see [22, pp. 281–282]. This says

$$\int_\Omega v(x,\,t)\cdot W(x)\,dx \longrightarrow \int_\Omega v(x,\,t_0)\cdot W(x)\,dx \tag{2.4}$$

for each $W \in L^2(\Omega)$ as $t \to t_0 \in [0, T]$. An easy consequence of (2.3) and (2.4) is the following form of the generalized energy inequality:

For $0 < t < T$ and for each smooth, compactly supported $\phi \geq 0$,

$$\begin{aligned} &\int_{\Omega\times\{t\}} |v|^2\,\phi\,dx + 2\int_0^t\!\!\int_\Omega |\nabla v|^2\,\phi\,dx\,dt \\ &\qquad \leq \int_0^t\!\!\int_\Omega \Big[|v|^2(\phi_t + \Delta\phi) + (|v|^2 + 2P)\,v\cdot\nabla\phi\Big]\,dx\,dt\,. \end{aligned} \tag{2.5}$$

In [1], the notion of (local) suitable weak solutions of (1.1) in an open set $D \subseteq \mathbb{R}^3 \times \mathbb{R}$ was introduced. A pair $(v,\,P)$ is a suitable weak solution of (1.1) in D if

(i) $P \in L^{3/2}(D)$ with $\int_D\int |P|^{3/2} \leq E$, and for some constants $E_0,\,E_1 < \infty$,

$$\int_{D_t} |v|^2\,dx \leq E_0\,, \quad D_t = D \cap (\mathbb{R}^3 \times \{t\})\,,$$

for a.e. t such that $D_t \neq \phi$ and

$$\int_D\!\!\int |\nabla v|^2\,dx\,dt \leq E_1\,;$$

(ii) (v, P) satisfies (1.1) in the sense of distributions on D; and

(iii) for each $0 \leq \phi \in C_0^\infty(D)$, inequality (2.3) is valid.

Note that we have added a pressure $L^{3/2}$-norm bounded condition here.

It is not at all clear if weak solutions obtained by the well-known Galerkin approximation procedure (see, for example, [22, chap. III, sec. 3]) are suitable weak solutions. That is why Caffarelli, Kohn, and Nirenberg used a different approach to show the existence of such suitable weak solutions in [1]. By using the estimate of Sohr and Von Wahl [20] for the pressure, one can show the following compactness result, which indicates the existence of such weak solutions. Indeed, our proofs of Lemma 2.3 and Theorem 2.2 below, along with the constructions in the appendix to [1], yield such suitable weak solutions.

THEOREM 2.2 *Let* (v_n, P_n) *be a sequence of weak solutions of* (1.1) *in* Q_1 *such that, for some positive constants* $E, E_0, E_1 < \infty$*, one has*

(a) $\displaystyle\int_{B_1\times\{t\}} |v_n|^2\,dx \leq E_0$ *for a.e.* $t \in (-1, 0)$;

(b) $\displaystyle\int_{Q_1}\!\!\int |\nabla v_n|^2\,dx\,dt \leq E_1$;

(c) $\int_{Q_1}\int |P_n|^{3/2}\,dx\,dt \le E$; *and*

(d) (v_n, P_n) *satisfies* (2.3) *for* $n = 1, 2, \ldots$.

Suppose that (v, P) *is the weak limit of* (v_n, P_n); *then* (v, P) *is a suitable weak solution of* (1.1) *on* Q_1.

Let us first prove the $L^{5/3}$ space-time norm estimates on P (cf. [20]). For simplicity we take $\Omega = Q_1$.

LEMMA 2.3 *Let* $(v,\, P)$ *be a weak solution of* (1.1)–(1.2) *in* Q_1 *with* $v \in L^\infty(1,0;H) \cap L^2(-1,0,V)$. *Then* $P \in L^{5/3}(-1,0;L^{5/3}(B_1))$.

PROOF: Let

$$q = \frac{30}{13}\,, \qquad a = \frac{3}{4}\left(\frac{30}{13} - 2\right) = \frac{3}{13}\,,$$

in the interpolation inequality (2.1). One obtains, for a.e. t, that

$$\begin{aligned}\|v\|^{30/31}_{L^{30/13}(B_1)} &\le c\left(\int_{B_1} |\nabla v|^2\,dx\right)^{3/13}\left(\int_B |v|^2\,dx\right)^{12/13} \\ &\quad + c\left(\int_{B_1}|v|^2 dx\right)^{15/13}. \end{aligned} \tag{2.6}$$

Thus by Hölder's inequality,

$$\|v\cdot\nabla v\|^{5/3}_{L^{15/14}(B_1)} \le c\left[\|\nabla v\|^2_{L^2(B_1)} + \|v\|^{10}_{L^{30/13}(B_1)}\right]. \tag{2.7}$$

Using (2.6), we have

$$\|v\|^{10}_{L^{30/13}(B_1)} \le c\left[\|v\|^2_{L^2(B_1)}\,\|v\|^8_{L^2(B_1)} + \|v\|^{10}_{L^2(B_1)}\right]. \tag{2.8}$$

Hence

$$v\cdot\nabla v \in L^{5/3}\left(-1,0;L^{15/14}(B_1)\right)\,. \tag{2.9}$$

Let $f = (\partial/\partial t)v - \Delta v$ in $B_1\times(-1,0)$; then $f \in L^2(-1,0,Z)$. Here Z is the dual space of $H^2_0(\Omega)$. Indeed, smooth, compactly supported $\phi \in C^\infty_0(B_1, R^3)$ with $\operatorname{div}\phi = 0$ is dense in both H and V. Also, since $v(t)$ is weak continuous in t with values in H, we may define a continuous function t, $(v(t), \phi)$, for ϕ as above. Then the claim follows from the estimate

$$\begin{aligned}\left|\left(\frac{\partial v}{\partial t}, \phi\right)\right| &= |-(\nabla v, \nabla\phi) - (v \cdot \nabla v, \phi)| \\ &\leq (\|\nabla v(t)\|_{L^2(B_1)} + \|v(t)\|_{L^2(B_1)} \|\nabla v(t)\|_{L^2(B_1)}) \|\phi\|_{H^2(B_1)} .\end{aligned}$$

Next, we observe, for a.e. t,

$$\begin{cases} \operatorname{div} f = 0 \\ \operatorname{curl} f = \operatorname{curl}(v \cdot \nabla v) \end{cases} \quad \text{in } B_1 . \tag{2.10}$$

Then the elliptic estimates ([14, chap. 7]) yield

$$\|f\|^{5/3}_{L^{15/14}(B_1)} \leq C \left[\|v \cdot \nabla v\|^{5/3}_{L^{15/14}(B_1)} + \|f\|^{5/3}_Z\right] . \tag{2.11}$$

Therefore, by integrating (2.11), one has

$$\frac{\partial}{\partial t} v - \Delta v \in L^{5/3}\left(-1, 0; L^{15/14}(B_1)\right),$$

and hence the conclusion of Lemma 2.3 follows from

$$\nabla P \in L^{5/3}\left(-1,\, 0; L^{15/14}\right)$$

and Sobolev's inequality. ∎

Proof of Theorem 2.2: We may assume that

$$v_n \to v$$

weakly in $L^2(-1, 0; V)$ and weak-$*$ in $L^\infty(-1, 0; H)$ and that $P_n \to P$ weakly in $L^{3/2}(Q_1)$. We wish to show that $v_n \to v$ strongly in $L^q(Q_1)$ for $1 \leq q < \frac{10}{3}$. Indeed, if the last statement is true, then for any smooth $\phi \geq 0$ compactly supported in Q_1, we have that

$$2 \int_{Q_1}\!\!\int |\nabla v|^2 \phi \, dx \, dt \leq \liminf_n 2 \int_{Q_1}\!\!\int |\nabla v_n|^2 \phi \, dx \, dt$$

by Fatou's lemma and that the right-hand sides of (2.3) (with v_n in the places of v) converge to the desired form as $v_n \to v$ strongly in $L^3(Q_1)$ and $P_n \to P$ weakly in $L^{3/2}_{\text{loc}}(Q_1)$. The result of Theorem 2.2 follows.

To show the strong convergence of v_n, we first establish the certain uniform weak continuity of v_n as functions of time t. In fact, we let Z be the dual of $H_0^2(B_1)$; then the equations

$$\frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n + \nabla P_n - \Delta v_n = 0 \quad \text{in } Q_1$$

and the fact that $v_n \to v$ weakly in $L^2(-1, 0; V)$, weak-$*$ in $L^\infty(-1, 0, H)$ along with $L^{3/2}$-norm bound for P_n imply that

$$\frac{\partial}{\partial t} v_n \in L^{3/2}(-1,\, 0;\, Z) \quad \text{with } \left\| \frac{\partial}{\partial t} v_n \right\|_{L^{3/2}(-1,\, 0;\, Z)} \le C_0$$

for some constant C_0 depending only on

$$\sup_n \Big[\|v_n\|_{L^2(-1,0;V)} + \|v_n\|_{L^\infty(-1,0;H)} + \|P_n\|_{L^{3/2}(Q_1)} \Big] .$$

Thus each $v_n \in C([-1,\, 0],\, Z)$. Moreover, they are uniformly continuous as functions of $t \in [-1, 0]$ with values in Z.

Now we apply theorem 2.1 of [22, chap. III] to conclude that the v_n stay in a compact set of $L^{3/2}(Q_1)$. Therefore $v_n \to v$ strongly in $L^{3/2}(Q_1)$. Since v_n also remains bounded uniformly in $L^{10/3}(Q_1)$, we deduce that $v_n \to v$ strongly in $L^q(Q_1)$ for all $1 \le q < \frac{10}{3}$. ∎

REMARK 2.4 Let (v_n, P_n) be as in Theorem 2.2. Then $v_n \to v$ weakly in $L^2(-1, 0; V)$ and weak-$*$ in $L^\infty(-1, 0; H)$. By the weak continuity of v in time, we see, for any $t_0 \in [-1,\, 0]$, $v(x,\, t_0)$ is a well-defined function in $L^2(B_1)$. We claim $v_n(x,\, t_0) \to v(x,\, t_0)$ weakly in $L^2(B_1)$ as $n \to \infty$. Indeed, for any $n_k \to \infty$, there is a subsequence on $\{v_{n_k}\}$ that converges weakly to $\widetilde{v}(x,\, t_0)$ for some $\widetilde{v}(x,\, t_0) \in L^2(B_1)$. It suffices to verify $\widetilde{v}(x,\, t_0) = v(x,\, t_0)$.

To do so we let $\phi(x) \in C_0^2(B_1)$ with $\operatorname{div} \phi = 0$, and let $\eta(t) \ge 0$ be smooth and compactly supported in a δ-neighborhood of $t_0 \in [-1,\, 0]$ with

$$\int_0^1 \eta(t) dt = 1 .$$

By the weak continuity of $v(\cdot,\, t)$ in t, we have

$$\int_{Q_1}\!\!\int \eta(t)\, v(x,\, t) \cdot \phi(x)\, dx\, dt = \int_{Q_1}\!\!\int \eta(t)\, v(x,\, t_0) \cdot \phi(x)\, dx\, dt + o(1) .$$

Here $o(1) \to 0$ as $\delta \to 0$.

One also has that

$$\int_{Q_1}\!\!\int \eta(t)\, v(x,\, t)\cdot\phi(x)\, dx\, dt = \int_{Q_1}\!\!\int \eta(t)\, v_{n_k}(x,\, t)\cdot\phi(x)\, dx\, dt + o(1)\,;$$

here $o(1) \to 0$ as $n_k \to \infty$.

Finally, by the uniform weak continuity of $v_n(\cdot, t)$ in t,

$$\int_{Q_1}\!\!\int \eta(t)\, v_{n_k}(x,\, t)\cdot\phi(x)\, dx\, dt = \int_{Q_1}\!\!\int \eta(t)\, v_{n_k}(x,\, t_0)\cdot\phi(x)\, dx\, dt + o(1)\,,$$

when $o(1) \to 0$ as $\delta \to 0$ (independently of n_k). We thus arrive at

$$\int_{B_1} v(x,\, t_0)\cdot\phi(x)\, dx = \int_{B_1} \widetilde{v}(x,\, t_0)\cdot\phi(x)\, dx\,.$$

Since $\phi \in C_0^2(B_1)$ with $\operatorname{div}\phi = 0$ is arbitrary, $v(x,\, t_0) = \tilde{v}(x,\, t_0)$.

3 Partial Regularity Theorem

Let $(v,\, P)$ be a suitable weak solution of (1.1) in Q_1.

THEOREM 3.1 *There are two positive constants ε_0 and C_0 such that*

$$\int_{Q_1}\!\!\int \left[|v|^3 + |P|^{3/2}\right] dx\, dt \le \varepsilon_0$$

implies

$$\|v(x,\, t)\|_{C^\alpha(Q_K)} \le C_0 \quad \textit{for some } \alpha > 0\,.$$

Here $Q_r = \{(x,\, t),\ |x| \le r,\ -r^2 \le t \le 0\}$.

To prove this theorem, we start with the following:

LEMMA 3.2 *Suppose that*

$$\int_{Q_1}\!\!\int \left[|v|^3 + |P|^{3/2}\right] dx\, dt \le \varepsilon_0$$

for some sufficiently small ε_0. Then

$$\begin{aligned} &\theta^{-5}\int_{Q_\theta}\!\!\int \left[\frac{|v - v_\theta|^3}{\theta^{\alpha_0}} + \frac{|P - P_\theta(t)|^{3/2}}{\theta^{\alpha_0}}\right] dx\, dt \\ &\quad\le \frac{1}{2}\int_{Q_1}\!\!\int \left[|v|^3 + |P|^{3/2}\right] dx\, dt \end{aligned} \tag{3.1}$$

for some positive constant θ *and* $\alpha_0 \in (0, \frac{1}{2})$ *where*

$$v_\theta = \theta^{-5} \iint_{Q_\theta} v(y, \tau) dy\, d\tau\,,$$
$$P_\theta(t) = \theta^{-3} \int_{B_\theta \times \{t\}} P(y, t)\, dy \quad \textit{for } -\theta^2 \le t \le 0\,.$$

PROOF: Suppose that Lemma 3.2 is false. Then there would be a sequence of weak solutions (v_i, P_i) with

$$\varepsilon_i = \|v_i\|_{L^3(Q_1)} + \|P_i\|_{L^{3/2}(Q_1)}$$

and such that (3.1) is not valid for (v_i, P_i). Let

$$u_i = \frac{v_i}{\varepsilon_i}\,, \qquad \widetilde{P}_i = \frac{P_i}{\varepsilon_i}\,;$$

then

$$\frac{\partial}{\partial t} u_i + \varepsilon_i u_i \nabla u_i + \nabla \widetilde{P}_i = \Delta u_i\,. \tag{3.2}$$

A simple computation also verifies that $(u_i, \widetilde{P}_i)$ is a suitable weak solution of (3.2). One also notices that

$$\Delta \widetilde{P}_i = -\varepsilon_i \frac{\partial u_i^k}{\partial x_l} \frac{\partial u_i^l}{\partial x_k} \quad \text{in } \mathcal{D}(Q_1)\,. \tag{3.3}$$

It follows from the generalized energy inequality (2.5) that the u_i's lie in a bounded set of $L^\infty(-1, 0; L^2_{\text{loc}}(B_1) \cap L^2(-1, 0; H^1_{\text{loc}}(B_1))$, and hence they lie in a bounded set of $L^{10/3}(-1, 0; L^{10/3}_{\text{loc}}(B_1))$ by (2.1).

Since $\widetilde{P}_i$ is bounded in $L^{3/2}(Q_1)$, it thus follows from the proof of Theorem 2.2 that u_i converges strongly (by taking subsequences if needed) in $L^3(-1, 0; L^3_{\text{loc}}(B_1))$. Let (u, P) be a weak limit of $(u_i, \widetilde{P}_i)$; then (3.2) implies that

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla P = \Delta u \\ \operatorname{div} u = 0 \end{cases} \qquad \text{in } Q_1\,. \tag{3.4}$$

By lower semicontinuity, one has that

$$\int_{Q_1}\!\!\int |u|^3\, dx\, dt \le 1\,, \qquad \int_{Q_1}\!\!\int |P|^{3/2}\, dx\, dt \le 1\,. \tag{3.5}$$

A simple estimate for the Stokes equation yields that u and P are smooth in the spatial variable and that u is Hölder continuous in the time variable with, say, exponent $2\alpha_0$. Thus, for suitable $\theta \in (0, \frac{1}{2})$, one has

$$\theta^{-5} \int_{Q_\theta}\int |u - u_\theta|^3 \, dx \, dt \leq \frac{1}{4}\theta^{\alpha_0} \,. \tag{3.6}$$

Since $u_i \to u$ strongly in $L^3(-1, 0; L^3_{\mathrm{loc}}(B_1))$, we have

$$\theta^{-5} \int_{Q_\theta}\int |u_i - u_{i,\theta}|^3 \, dx \, dt \leq \frac{1}{3}\,\theta^{\alpha_0} \quad \text{for all sufficiently large } i \,. \tag{3.7}$$

Next we consider $\widetilde{P}_i$. By (3.3), we may write, for a.e. $t \in (-1, 0)$, that

$$\widetilde{P}_i(x,\, t) = h_i(x,\, t) + g_i(x, t)\,, \quad x \in B_{2/3}\,. \tag{3.8}$$

Here

$$\begin{cases} \Delta g_i(\cdot, t) = -\varepsilon_i\, \partial u_i^k/\partial x_l \partial u_i^l/\partial x_k & \text{in } B_{2/3} \\ g_i(\,\cdot\,,\, t) = 0 & \text{on } \partial B_{2/3}\,. \end{cases} \tag{3.9}$$

Hence $h(\cdot, t)$ is harmonic in $B_{2/3}$.

Let

$$\widetilde{P}_{i,\theta}(t) = \theta^{-3} \int_{B_\theta} h_i(x, t)\, dx \equiv h_{i,\theta}(t)\,;$$

then

$$\begin{aligned} \int_{Q_\theta}\int |\widetilde{P}_i - \widetilde{P}_{i,\theta}|^{3/2} \, dx \, dt &\leq C_0 \int_{Q_\theta}\int |h_i - h_{i,\theta}(t)|^{3/2}\,, \\ C_0 \int_{Q_\theta}\int |g_i|^{3/2} \, dx \, dt &\leq C_0\theta^5 \cdot \theta^{3/2} + C_0\varepsilon_i \int_{Q_{3/2}}\int |u_i|^3 \, dx \, dt \,. \end{aligned} \tag{3.10}$$

Here we have used the Calderon-Zygmund estimate for g_i by (3.9). The latter also implies that h_i is bounded in $L^{3/2}(-1, 0; L^{3/2}(B_{2/3}))$ as both $\widetilde{P}_i$ and g_i are. Thus the first term in the right-hand side of (3.10) follows from the interior estimate for harmonic functions.

It is obvious from (3.10) that

$$\theta^{-5} \int_{Q_\theta}\int |\widetilde{P}_i - \widetilde{P}_{i,\theta}|^{3/2} \, dx \, dt \leq \frac{1}{3}\,\theta^{\alpha_0} \tag{3.11}$$

for a suitable positive $\theta \in (0, \frac{1}{2})$ and for all sufficiently large i.

Combining (3.11) and (3.7) we obtain a contradiction. Thus the lemma is proven. ∎

PROOF OF THEOREM 3.1: Let $(v,\, P)$ be a suitable weak solution such that

$$\int_{Q_1}\!\!\int \left[|v|^3 + |P|^{3/2}\right] dx\, dt \le \varepsilon_0\,.$$

Let

$$\begin{aligned} v_1(x,\, t) &= \frac{v - v_\theta}{\theta^{\alpha_0/3}}\,(\theta x,\, \theta^2 t)\,,\\ P_1(x,\, t) &= \theta^{1-\alpha_0/3}\Big(P(\theta x,\, \theta^2 t) - P_\theta(t)\Big)\,. \end{aligned}$$

Again, a direct computation yields that $(v_1,\, P_1)$ is a suitable weak solution of

$$\frac{\partial v_1}{\partial t} + \theta(v_\theta + \theta^{\alpha_0/3} v_1)\cdot \nabla v_1 + \nabla P_1 = \Delta v_1 \quad \text{in } Q_1\,. \tag{3.12}$$

Moreover, Lemma 3.2 implies that

$$\int_{Q_1}\!\!\int \left[|v_1|^3 + |P_1|^{3/2}\right] dx\, dt \le \frac{\varepsilon_0}{2}\,. \tag{3.13}$$

We repeat the same arguments as in the proof of Lemma 3.2 except that (3.4) is replaced by

$$\begin{cases} \frac{\partial u}{\partial t} + \vec{b}\cdot \nabla u + \nabla P = \Delta u \\ \operatorname{div} u = 0 \end{cases} \qquad \text{in } Q_1\,. \tag{3.14}$$

Here $\vec{b} = \lim_i \theta v_{i,\,\theta}$ is a constant with $|\vec{b}| \le 1$. Note that $v_{i,\,\theta} \to 0$ as $\varepsilon_i \to 0$ (cf. the proof of Lemma 3.2).

Therefore we conclude that

$$\begin{aligned} &\theta^{-5}\int_{Q_\theta}\!\!\int \left[\frac{|v_1 - v_{1,\theta}|^3}{\theta^{\alpha_0}} + \frac{|P_1 - P_{1,\theta}|^{3/2}}{\theta^{\alpha_0}}\right] dx\, dt \\ &\qquad \le \frac{1}{2}\int_{Q_1}\!\!\int \left[|v_1|^3 + |P_1|^{3/2}\right] dx\, dt \le \frac{1}{4}\varepsilon_0\,. \end{aligned} \tag{3.15}$$

By a simple iteration, we then conclude that

$$r^{-5}\int_{Q_r}\!\!\int |v - v_r|^3\, dx\, dt \le C\varepsilon_0 r^{\alpha_0} \tag{3.16}$$

for all $r \in (0, \frac{1}{2})$. Thus v is Hölder continuous in $(x,\, t)$. The conclusion of Theorem 3.1 follows. ∎

The main result of [1] can be deduced from the following theorem (cf. also [12]):

THEOREM 3.3 *There is a positive constant ε_0 such that if*

$$\limsup_{r\to 0} r^{-1}\int_{Q_r}\int |\nabla v|^2\,dx\,dt \le \varepsilon_0\,,$$

then there are $\theta_0, r_0 \in (0,1)$ such that either

$$A^{3/2}(\theta_0 r) + D^2(\theta_0 r) \le \frac{1}{2}\left(A^{3/2}(r) + D^2(r)\right)$$

or

$$A^{3/2}(r) + D^2(r) \le \varepsilon_1 \ll 1$$

where $0 < r < r_0$, and

$$A(r) = \sup_{-r^2\le t\le 0} r^{-1}\int_{B_r\times\{t\}} |v|^2\,dx\,, \qquad D(r) = r^{-2}\int_{Q_r}\int |P|^{3/2}\,dx\,dt\,.$$

PROOF OF THEOREM 3.3: We also define

$$B(r) = r^{-1}\int_{Q_r}\int |\nabla v|^2\,dx\,dt\,, \qquad C(r) = r^{-2}\int_{Q_r}\int |v|^3\,dx\,dt\,.$$

Here $\theta_0 \in (0, \frac{1}{4})$, which will be chosen later. ∎

First we have Lemma 2.1, which says

$$C(r) \le C_0\left[\left(\frac{r}{\rho}\right)^3 A(\rho)^{3/2} + \left(\frac{\rho}{r}\right)^3 A(\rho)^{3/4}B(\rho)^{3/4}\right] \tag{3.17}$$

for $0 < r < \rho$ (cf. [1, lemma 5.2]). Next, we want to control $D(r)$.

LEMMA 3.4 *Let (v, p) be a weak solution of* (1.1) *in Q_1. Then, for almost all $t \in (-\frac{1}{2}, 0)$, one has*

$$\int_{B_\theta\times\{t\}} |p|^{3/2} \le C_{\theta_0}\int_{B_1\times\{t\}} |v-\bar v|^3\,dx + C_0\int_{B_1\times\{t\}} |p|^{3/2}\,dx$$

for all $\theta \in (\theta_0, \frac{1}{4})$.

PROOF: Since, for a.e. $t \in (-\frac{1}{2}, 0)$, one has $\Delta p = \partial_{x_i} v^j \partial_{x_j} v^i$ in $\mathcal{D}'(B_1)$, we may write $p = p_0 + h$ in B_ρ. Here $\rho \in (\frac{1}{2}, 1)$ is chosen so that

$$\int_{\partial B_\rho} |p|^{3/2} \, d\sigma \leq 3 \int_{B_1} |p|^{3/2} \, dx \,,$$

where

$$\begin{cases} \Delta p_0 = \partial_{x_i} (v^j - \bar{v}^j) \, \partial_{x_j} (v^i - \bar{v}^i) & \text{in } B_\rho \,, \\ p_0 = 0 & \text{on } \partial B_\rho \,, \end{cases}$$

and

$$\begin{cases} \Delta h = 0 & \text{in } B_\rho \,, \\ h = p & \text{on } \partial B_\rho \,. \end{cases}$$

Thus

$$\left(\int_{B_\theta \times \{t\}} |p|^{3/2} \, dx \right)^{3/2} \leq \left(\int_{B_\theta \times \{t\}} |p_0|^{3/2} \, dx \right)^{2/3} + \left(\int_{B_\theta \times \{t\}} |h|^{3/2} \, dx \right)^{2/3} .$$

The first term on the right-hand side of the last inequality is bounded by

$$\left(C_{\theta_0} \int_{B_1 \times \{t\}} |v - \bar{v}|^3 \, dx \right)^{2/3}$$

by Calderon-Zygmond's estimate. The second term can be bounded by

$$C_0 \int_{B_1 \times \{t\}} |p|^{3/2} \, dx$$

due to subharmonicity of $|h|^{3/2}$ in B_ρ and our choice of $\rho \in (\frac{1}{2}, 1)$. ∎

COROLLARY 3.5 *For any* $r \in (\theta_0 \rho, \frac{\rho}{2})$, $\rho \leq 1$, *one has*

$$\frac{1}{r^2} \int_{B_r \times \{t\}} |p|^{3/2} \, dx \leq C_{\theta_0} \frac{1}{\rho^2} \int_{B_\rho \times \{t\}} |v - \bar{v}|^3 \, dx + C_0 \left(\frac{r}{\rho} \right) \frac{1}{\rho^2} \int_{B_\rho} |p|^{3/2} \, dx \,.$$

Thus

$$D(r) \leq C_{\theta_0} \frac{1}{\rho^2} \int_{Q_p} \!\! \int |v - \bar{v}_\rho|^3 \, dx \, dt + C_0 \left(\frac{r}{\rho} \right) D(\rho) \,. \tag{3.18}$$

Next we use $|v|^2 - |\bar{v}|^2$ instead of $|v|^2$ in the generalized energy inequality (2.5). Here

$$|\bar{v}|^2(t) = \int_{B_\rho \times \{t\}} |v|^2 \, dy \,.$$

By using Poincaré's inequality

$$\left(\int_{B_\rho\times\{t\}} \left(|v|^2 - |\bar{v}|^2\right)^{3/2} dx\right)^{2/3} \le c\rho \int_{B_\rho\times\{t\}} |v|\,|\nabla v|\,dx$$

and then applying Hölder's inequality in the integration with respect to time t, one obtains from (2.5) for a properly chosen cutoff function ϕ that

$$\begin{aligned} A(r) + B(r) \le C \Bigg[&\frac{\rho}{r} C(\rho)^{2/3} + \frac{\rho^{2/3}}{r} C(\rho)^{1/3} A(\rho)^{1/2} B(\rho)^{1/2} \\ &+ \cdots + \frac{\rho}{r} C(\rho)^{1/3} D(\rho)^{2/3} \Bigg]. \end{aligned} \tag{3.19}$$

Similarly, by using Poincaré's inequality, one obtains from (3.18) that

$$D(r) \le C\left[\frac{r}{\rho} D(\rho) + \left(\frac{\rho}{r}\right)^2 B(\rho)^{3/4} A(\rho)^{3/4}\right]. \tag{3.20}$$

By combining (3.19) with $\rho = 2r$, (3.17), and (3.20), one may easily deduce that

$$A(\theta_0 r)^{3/2} + D^2(\theta_0 r) \le C_1\theta_0 \left(A(r)^{3/2} + D^2(r)\right) + \varepsilon_1$$

for $r \le r_0$. Here C_1 is a constant independent of θ_0 and ε_1 is a constant depending only on certain powers of θ_0^{-1} and $B(r)$. By choosing r_0 small enough, we may assume ε_1 is also very small. The conclusion of Theorem 3.3 follows. We finally note that the regularity of v at certain points where the hypothesis of Theorem 3.3 is satisfied follows from the conclusions of Theorem 3.1 and the decay estimates of Theorem 3.3.

4 Final Remarks

Let $(v,\, P)$ be a weak solution of (1.1); then $(v_\lambda,\, P_\lambda)$ is also a weak solution of (1.1) for all $\lambda > 0$. Here

$$v_\lambda(x,\, t) = \lambda v(\lambda x,\, \lambda^2 t)\,, \qquad P_\lambda(x,\, t) = \lambda^2 P(\lambda x,\, \lambda^2 t)\,.$$

In other words, v is of dimension -1 and P is of dimension -2. For Leray-Hopf solutions, one has two basic estimates:

(a) $\iint \left[|v|^{10/3} + |P|^{5/3}\right] dx\,dt < \infty$ and

(b) $\iint |\nabla v|^2\,dx\,dt < \infty$.

Since

$$\int_{Q_r}\int \left[|v|^{10/3} + |P|^{5/3}\right] dx\, dt$$

is of dimension $\frac{5}{3}$, Theorem 3.1 says the singular set of suitable weak solutions is of measure zero with respect to parabolic Hausdorff measure $P^{5/3}$. Similarly, Theorem 3.3 implies the singular set of P^1 measure zero.

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