5.8 Regularity

5.8.1 Scaling and Dimension Analysis

Suppose (v, p) solves (NS) with force f in $Q_r \coloneqq B_r \times (0, r^2)$, we scaling it as

$$v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), f^\lambda(x,t) = \lambda^3 f(x,t).$$

Then (v^λ, p^λ) solves (NS) with force f^λ in $Q_{r/\lambda}$. Indeed, we see

$$\begin{aligned}
&\left(\partial_t v^\lambda - \Delta v^\lambda + v^\lambda \cdot \nabla v^\lambda + \nabla p^\lambda\right)(x,t) \\
=&\partial_t\left(\lambda v(\lambda x, \lambda^2 t)\right) - \Delta(\lambda v(\lambda x, \lambda^2 t)) \\
&+ \lambda v(\lambda x, \lambda^2 t) \cdot \nabla(\lambda v(\lambda x, \lambda^2 t)) + \nabla\left(\lambda^2 p(\lambda x, \lambda^2 t)\right) \\
=&\lambda^3\left(\partial_t v - \Delta v + v \cdot \nabla v + \nabla p\right)(\lambda x, \lambda t) \\
=&\lambda^3 f(\lambda x, \lambda^2 t) = f^\lambda(x,t)
\end{aligned}$$

for $(x,t) \in Q_{r/\lambda}$. A quantity $\varphi(r, v, p, f)$ is ***of dimension*** d if

$$\varphi(r, v, p, f) = r^d \varphi(1, v^r, p^r, f^r).$$

Especially, φ is called ***homogenous*** if it is of dimension 0.

Proposition 5.22. *Let v a very weak solution of* (NS) *in Q_1 with $f \in C^\infty$ and*

$$v \in L^\infty L^2 \cap L^2 H^1 \cap L^s L^q(Q_1)$$

with $3/q + s/2 \le 1, q > 3$, then $v \in L^\infty(Q_{1/2})$.

Proof. Indeed, we notice that

$$\|v\|_{L^\infty(Q_{1/2})} \lesssim \|\nabla v\|_{L^\infty L^4(Q_{1/2})}{}^{1} \lesssim \|\nabla \times v\|_{L^\infty L^4(Q_R)} + \|\nabla \cdot v\|_{L^\infty L^4(Q_R)} + \|v\|_{L^\infty L^1(Q_R)}$$

for any $0 < 1/2 < R \le 1$, by Poincare embedding and elliptic estimate. As $\nabla \cdot v = 0$ and $\|v\|_{L^\infty L^1} \lesssim \|v\|_{L^\infty L^2}$, the remained task is to give a $L^\infty L^4$-estimate for the vorticity $w = \nabla \times v$. Now we consider the vorticity equation:

$$\begin{aligned}
&\partial_t w - \Delta w = \nabla \times (v \cdot \nabla v) = w \cdot \nabla v - v \cdot \nabla w \\
\Longrightarrow&\partial_t w^i - \Delta w^i = w^j \partial_j v^i - v^j \partial_j w^i = \partial_j(w^j v^i - v^j w^i) \\
\Longrightarrow&(\partial_t - \Delta) w = \nabla \cdot g, g^{ij} = w^j v^i - v^j w^i.
\end{aligned}$$

We already have $w \in L^2(Q_1)$ since $v \in L^2 H^1$, and the goal is to reach $w \in L^\infty(Q_R)(\hookrightarrow L^\infty L^4(Q_R))$ for some $R \in (1/2, 1)$ by bootstrap on integrable index and radius. Set the step-length δ, steps K, step radius r as

$$\delta \le 1 - \frac{3}{q} - \frac{2}{s}, K\delta = \frac{1}{2}, r^{2K+1} = \frac{1}{2},$$

[1] 4 is the smallest p such that $1 - \frac{3}{p} > -\frac{3}{\infty}$.

and bootstrap:

$$p_0 = 2, \frac{1}{p_k} - \frac{1}{p_{k+1}} = \delta(\text{so } p_K = \infty);$$
$$r^0 = 1, \cdots, r^{2K}(=: R) > \frac{1}{2}.$$

Suppose $w \in L^{p_k}(Q_{r^{2k}})$ for some $k \leq K-1$, now we claim $w \in L^{p_{k+1}}(Q_{r^{2(k+1)}})$. We decomposition $w = \tilde{w}_k + h_k$ as

$$(\partial_t - \Delta)\tilde{w} = \nabla \cdot (\zeta_k g), \text{ where } \zeta_k(x,t) = \zeta_k(r^{-2k}x, r^{-4k}t), \zeta \in \mathcal{D}(Q_1) \text{ cutoff on } Q_r,$$

so $(\partial_t - \Delta)h = 0$ on $Q_{r^{2k+1}}$. Then the interior estimate for heat equation implies $h \in L^\infty(Q^{2(k+1)})$. As the L^p-estimate for $\tilde{w} = \Phi *_{x,t} (\nabla \cdot (\zeta_k g)) = \nabla \Phi * (\zeta_k g)$[1], remind

$$\|\nabla \Phi(t)\|_\beta \lesssim t^{-(2-\frac{3}{2\beta})} \Longrightarrow \|\nabla \Phi\|_{\alpha,\beta} \lesssim 1, \text{ if } \alpha(2 - \frac{3}{2\beta}) < 1; \tag{5.13}$$

so Young's and Holder's interpolation gives:

$$\begin{aligned}\|\tilde{w}\|_{L^{p_{k+1}}_{t,x}} \lesssim & \|\nabla \Phi * (\zeta_k g)\|_{p_{k+1},p_{k+1}} \lesssim \|\nabla \Phi\|_{\alpha,\beta} \|\zeta_k g\|_{a,b} \\ \leq & \|\nabla \Phi\|_{\alpha,\beta} \|v\|_{s,q} \|w\|_{L^{p_k}(Q_{r^{2k}})}\end{aligned}$$

if we can find α, β, a, b satisfies (5.13) and

$$\begin{cases} \frac{1}{p_{k+1}} = \frac{1}{\alpha} + \frac{1}{a} - 1 = \frac{1}{\beta} + \frac{1}{b} - 1; \\ \frac{1}{a} = \frac{1}{s} + \frac{1}{p_k}, \frac{1}{b} = \frac{1}{q} + \frac{1}{p_k}, \\ \frac{2}{s} + \frac{3}{q} = 1. \end{cases}$$

This can be reach by the selection of δ, K, r.▲ □

5.8.2 $\epsilon-$regularity

Now we give a significant compactness method to verify the $\epsilon-$regularity criterion. Consider the following quantities:

$$\begin{aligned} &\phi(z,r,v) = r|v_{Q_{z,r}}|, \psi_{\beta,\gamma}(z,r,f) = r^{1+\beta}\|f\|_{L^{2,2\gamma+1}(Q_{z,r})}; \\ &\psi(z,r,v) = r^{-\frac{2}{3}}\|v\|_{L^3(Q(z,r))}, \psi(z,r,p) = r^{-\frac{4}{3}}\|p\|_{L^{\frac{3}{2}}(Q(z,r))}; \\ &\varphi(z,r,v) = \left(\frac{1}{r^2}\int_{Q(z,r)} \left|v - (v)_{Q(z,r)}\right|^3\right)^{\frac{1}{3}} = r^{-\frac{2}{3}}\|v - (v)_{z,r}\|_{L^3(Q(z,r))}, \\ &\varphi(z,r,p) = \left(\frac{1}{r^2}\int_{Q(z,r)} \left|p - (p)_{B(x,r)}\right|^{\frac{3}{2}}\right)^{\frac{2}{3}} = r^{-\frac{4}{3}}\|p - (p)_{z,r}\|_{L^{\frac{3}{2}}(Q(z,r))}. \end{aligned} \tag{5.14}$$

[1] $\Phi(x,t)$ is the heat kernel.

For convenience, we'd like to omit z if $z=0$, and denote $\varphi(r,v,p) \coloneqq \varphi(r,v)+\varphi(r,p)$, etc. It is easy to check those dimensions:

$$\begin{aligned}
&\phi(r,v) = \phi(1,v^r),\\
&\psi_{\beta,\gamma}(r,f) = r^{-(\gamma-\beta)}\psi_{\beta,\gamma}(1,f^r),\\
&\psi/\varphi(r,v,p) = \psi/\varphi(1,v^r,p^r);
\end{aligned}$$

Proposition 5.23. $\|(u)_{B_r}\|_{L^q(B_r)} \lesssim \|u\|_{L^q(B_r)}$.

Some facts for those quantities are important:

Proposition 5.24. *1.* $\|w\|$

The estimate below is necessary for the following iteration.

Lemma 5.25. $\exists C>0, \forall 0<\beta,\gamma\le 2, \theta\in(0,\frac{1}{2}], \exists\epsilon_2, r_2\in(0,1]$ *such that if* $\exists r\le r_2$, (v,p,f) *is a suitable weak solution of* (NS) *in* Q_r *and*

$$\phi(r,v)\le 1, \varphi(r,v,p)+\psi(r,f)\le\epsilon_2,$$

then $\phi(\theta r,v)\le 1$ *and* $\varphi(\theta r,v,p)\le C\theta^{1+\frac{2}{3}}\left(\varphi(r,v,p)+\psi(r,f)\right)$.

Remark. *Notice that the result can be generalized to* $z\neq 0$ *case.*

Proof. The first claim is clear:

$$\begin{aligned}
\phi(\theta r,v) =&\theta r\cdot|Q_{\theta r}|^{-1}\int_{Q_{\theta r}}(v-v_r)+\theta r v_r\\
\le& C'\theta r\cdot|Q_{\theta r}|^{1-\frac{1}{3}-1}\|v-v_r\|_{L^3(Q_{\theta r})}+\theta r v_r\\
\le& C'\theta^{-\frac{2}{3}}\varphi(r,v)+\theta\phi(r,v)\\
\le& C'\epsilon_2\theta^{-\frac{2}{3}}+\frac{1}{2}\le 1
\end{aligned}$$

if $\epsilon_2\le\theta^{\frac{2}{3}}/(2C')$. As for the latter, we prove it by contradiction: if not, then $\forall C>0, \exists\beta<\gamma, \theta\in(0,\frac{1}{2}], \forall\epsilon_2, r_2>0$(where we set both as $\dfrac{1}{n}$), if $\exists r_n\le\dfrac{1}{n}$, (v_n,p_n) are SWS of (NS) in Q_{r_n} such that

$$\psi(r_n,v_n)\le 1, \epsilon_n\coloneqq\varphi(r_n,v_n,p_n)+\psi(r_n,f_n)\le\frac{1}{n},$$

then $\varphi(\theta r_n,v_n,p_n) > C\theta^{1+\frac{2}{3}}\left(\varphi(r_n,v_n,p_n)+\psi(r_n,f_n)\right) = C\epsilon_n\theta^{1+\frac{2}{3}}$. Before the contradiction, first we take a scaling by following formulation:

$$(b_n,u_n,q_n,g_n)\coloneqq\left((v_n^{r_n})_1, \epsilon_n^{-1}(v_n^{r_n}-(v_n^{r_n})_1), \epsilon_n^{-1}(p_n^{r_n}-(p_n^{r_n})_1), \epsilon_n^{-1}f_n^{r_n}\right).$$

Then the homogeneity of ψ and φ transform the above relations as

$$|b_n| \leq 1, \varphi(1, u_n, q_n) + r_n^{-(\gamma-\beta)}\psi(1, g_n) \overset{1}{=} \|u_n\|_3 + \|q_n\|_{\frac{3}{2}} + r_n^{-(\gamma-\beta)}\|g_n\|_{2,2\lambda+1} = 1, \quad (5.15)$$

and the claim: $\forall C > 0, \varphi(\theta, u_n, q_n) \geq C\theta^{1+\frac{2}{3}}\left(\varphi(1, u_n, q_n) + r_n^{-(\gamma-\beta)}\psi(1, g_n)\right) = C\theta^{1+\frac{2}{3}}$. The remained work is to show $\varphi(\theta, u_n), \varphi(\theta, q_n) \lesssim \theta^{1+\frac{2}{3}}$, which counter the claim immediately.

1. $\varphi(\theta, u_n) \lesssim \theta^{1+\frac{2}{3}}$: Notice (u_n, q_n) is suitable weak solution of following system on Q_1:

$$\begin{cases} \partial_t u_n - \Delta u_n + (b_n + \epsilon_n u_n) \cdot \nabla u_n + \nabla q_n = g_n, \\ \nabla \cdot u_n = 0. \end{cases}$$

Moreover, the uniform bound (5.15) implies

$$b_n \xrightarrow{\mathbb{R}} b, u_n \xrightharpoonup{L^3} u, q_n \xrightharpoonup{L^{\frac{3}{2}}} q, g_n \xrightarrow{L^{2,2\gamma+1}} 0.$$

Now we attempt to derive a strong L^q-convergence for u_n by Aubin-Lion lemma: The local energy inequality implies[2]

$$\|u_n\|_{L^{10/3}(Q_{7/8})} \lesssim \|u_n\|_{L^\infty L^2 \cap L^2 \dot{H}^1(Q_{7/8})} \lesssim 1,$$

and following calculation implies $\|\partial_t u_n\|_{L^{\frac{4}{3}}(H^1_\sigma)'} \lesssim 1$:

$$\begin{aligned} \left|\int \partial_t u_n \cdot \zeta\right| \leq & \left|\int \nabla u_n \cdot \nabla\zeta\right| + \left|\int \nabla u_n \cdot \zeta\right| + \left|\int u_n u_n \nabla\zeta\right| \\ \lesssim & \|u_n\|_{L^2\dot{H}^1}\|\zeta\|_{\dot{H}^1} + \|u_n\|_{L^2\dot{H}^1}\|\zeta\|_{L^2} + \|u_n\|_{L^\infty L^2}^{\frac{1}{2}}\|u_n\|_{L^2L^6}^{\frac{3}{2}}\|\nabla\zeta\|_{L^4L^2} \\ \lesssim & \|\zeta\|_{L^4(H^1_\sigma)}. \end{aligned}$$

Since $H^1 \overset{K}{\hookrightarrow} L^2 \hookrightarrow (H^1_\sigma)'$, together with $\|u_n\|_{L^2H^1} \lesssim 1$ and $\|\partial_t u_n\|_{L^{\frac{4}{3}}(H^1_\sigma)'} \lesssim 1$, we see $\{u_n\}$ is precompact in L^2L^2 and thus $u_n \xrightarrow{L^2} u$. Moreover, since $\|u_n\|_{L^{10/3}} \lesssim 1$, so the Holder interpolation implies[3] $u_n \xrightarrow{L^q(Q_{7/8})} u, \forall q \in [2, 10/3)$.
Consequently, (u, q) solves[4] following system in $Q_{\frac{7}{8}}$:

$$\begin{cases} \partial_t u - \Delta u + b \cdot \nabla u + \nabla q = 0, \\ \nabla \cdot u = 0. \end{cases}$$

---

[1] $u_n = n(v_n^{r_n} - (v_n^{r_n})_1) = n(v_n^{r_n} - (v_n^{r_n})_1) - n((v_n^{r_n} - (v_n^{r_n})_1)_1) = u_n - (u_n)_1$. Similar for q_n.
[2] **?**is the initial data for each u_n determined?
[3] $\|u_m - u_n\|_q \leq \|u_m - u_n\|_2^{\theta}\|u_m - u_n\|_{10/3}^{1-\theta} \lesssim \|u_m - u_n\|_2^{\theta}, \forall q = \frac{\theta}{2} + \frac{1-\theta}{10/3}$.
[4] Particularly, it is a suitable weak solution which is preseved by weak limit. Details see Lin's Paper, Theorem 2.2.

Following we show that u is Holder continuous:

$$(\partial_t - \Delta + b\cdot\nabla)(\nabla\times u) = 0 \Longrightarrow \nabla\times u \in L^\infty(Q_{5/6}) \xRightarrow{\nabla\cdot u=0} \nabla u \in L^\infty L^{100}(Q_{4/5}),$$
$$q\in L^{\frac{3}{2}}, \Delta q(t) = 0 \Longrightarrow \nabla q \in L^{3/2}C^\infty \xRightarrow{\nabla u\in L^\infty L^{100}, \Delta u \in ?} \partial_t u \in L^{3/2}L^\infty(Q_{4/5}).$$

It comes $u\in C^{\alpha,\frac{\alpha}{2}}(\overline{Q_{3/4}})$ for $\frac{\alpha}{2} = 1 - \frac{1}{3/2} \Longrightarrow \alpha = \frac{2}{3}$. Then the Campanato characterization in parabolic version gives out following bound since $\theta \le 3/4$:

$$\theta^{-3-2-3\alpha}\int_{Q_\theta} |u-(u)_\theta|^3 \lesssim 1 \xRightarrow{u_n \xrightarrow{L^3} u} \theta^{-3-2-3\alpha}\int_{Q_\theta}|u_n - (u_n)_\theta|^3 \lesssim 1,$$

which implies $\varphi(\theta,u)\lesssim \theta^{1+\alpha}$.

2. $\varphi(\theta,q_n)\lesssim 1$: notice $\Delta q_n = \epsilon_n \nabla\cdot(u_n\cdot\nabla u_n) + \nabla\cdot g_n = \epsilon_n(\partial_i\partial_j(v^iv^j)) + \nabla\cdot g_n$ on $Q_{7/8}$, so we split $q_n = \tilde{q}_n + h_n$ where

$$\Delta\tilde{q}_n = \epsilon_n\zeta(\partial_i\partial_j(v^iv^j)) + \zeta\nabla\cdot g_n, \zeta\in D(Q_{\frac{7}{8}}) \text{ cutoff on } Q_{\frac{3}{4}};$$
$$\Delta h_n = \Delta(q_n - \tilde{q}_n) = 0 \text{ on } Q_{\frac{3}{4}}.$$

We estimate $\tilde{q}_n$ by Riesz potential, and h_n by properties of harmonic function:

$$\begin{aligned}\|\tilde{q}_n\|_{L^3(Q_{7/8})} =& \left\|I^2\left((\epsilon_n\zeta(\partial_i\partial_j(u_n^iu_n^j)) + \zeta\nabla\cdot g_n)\right)\right\|_{L^{3/2}(Q_{7/8})}\\ \lesssim&\epsilon_n\left\|\partial^2\Gamma * (u_n^iu_n^j)\right\|_{L^{3/2}(Q_{7/8})} + {}^1\|g_n\|_{L^{\frac{3}{2}}L^2(Q_{7/8})}\\ \lesssim&\epsilon_n\|u_n\|^2_{L^3(Q(7/8))} + \|g_n\|_{L^{2,2\gamma+1}(Q_{7/8})} \lesssim \epsilon_n + r_n^{\gamma-\beta};\end{aligned}$$

$$\begin{aligned}\int_{Q_\theta}|h_n - (h_n)_\theta|^{\frac{3}{2}} =& \int_{-\theta^2}^0\int_{B_\theta}|h_n-(h_n)_\theta|^{\frac{3}{2}} = \int_{-\theta^2}^0\int_{B_\theta}(\theta\|h_n\|_{lip(Q_\theta)})^{\frac{3}{2}}\\ \lesssim&\theta^{3+\frac{3}{2}}\int_{-9/16}^0\|\partial h_n\|^{\frac{3}{2}}_{L^\infty(B_{3/4})} \lesssim \theta^{3+\frac{3}{2}}\int_{-9/16}^0\|h_n\|^{3/2}_{L^1(B_{3/4})}\\ \lesssim&\theta^{3+\frac{3}{2}}\int_{-9/16}^0\|h_n\|^{3/2}_{L^{3/2}(B_{3/4})} \lesssim \theta^{3+\frac{3}{2}}\int_{Q_{3/4}}|h_n|^{3/2}\\ \lesssim&\theta^{3+\frac{3}{2}}\int_{Q_{3/4}}\left(|q_n|^{3/2} + |\tilde{q}_n|^{3/2}\right) \lesssim \theta^{3+\frac{3}{2}}.\end{aligned}$$

---

[1] this is reached by following estimate:

$$\left\|I^2(\zeta\nabla\cdot g_n)\right\|_{L^{\frac{3}{2}}(B_{7/8})} \lesssim \left\|I^2\nabla(\zeta g_n) + I^2 g_n\right\|_{3/2}$$
$$\lesssim \left\|I^1 g_n\right\|_6 + \left\|I^2 g_n\right\|_{\frac{15}{2}} \lesssim \|g_n\|_2 + \|g\|_{5/4} \le \|g_n\|_2.$$

Here we will encounter critical case $q=1$ for $\|g\|_q$ if no amplification by cutoff function.

And then the estimate q_n follows:

$$\int_{Q_\theta} |q_n - (q_n)_\theta|^{\frac{3}{2}} \lesssim \int_{Q_\theta} |h_n - (h_n)_\theta|^{\frac{3}{2}} + \int_{Q_\theta} |\tilde{q}_n - (\tilde{q}_n)_\theta|^{\frac{3}{2}}$$
$$\lesssim \theta^{3+\frac{3}{2}} + \left(\epsilon_n + r_n^{\gamma-\beta}\right)^{3/2} \lesssim \theta^{3+\frac{3}{2}}$$

if $\epsilon_n + r_n^{\gamma-\beta} \lesssim \theta^3$. Consequently, we have $\varphi(\theta, q_n) \lesssim \theta^{1+\frac{2}{3}}$.

Combine together, we get $\varphi(\theta, u_n, p_n) \lesssim \theta^{1+\frac{2}{3}}$. □

Proposition 5.26. *For any $\gamma \in (0, 2], \alpha \in \min\{\frac{2}{3}, \gamma\}$, there are $\epsilon_1 > 0, \theta \in (0, \frac{1}{2}), r_1 > 0$ such that if $\exists R \leq r_1, f \in L^{2,2\gamma+1}(Q_R), (v, p, f)$ is a suitable weak solution of (NS) in Q_R, and*

$$\psi(R, v, p) \leq \epsilon_1.$$

Then $v \in C^{\alpha,\frac{\alpha}{2}}(Q_{\theta R})$.

Proof. First we set following relations to apply the above lemma:

$$\beta = \frac{1}{2}(\alpha + \gamma); \theta \in (0, \frac{1}{2}], C\theta^{\frac{2}{3}} + \theta^\beta \leq \theta^\alpha;$$
$$(\epsilon_2, r_2) \sim (\theta, \beta, \gamma) \sim (\gamma, \alpha);$$
$$r_1 = \min\left\{ r_2, \left(\frac{\epsilon_2}{2\|f\|_{L^{2,2\lambda+1}(Q_R)}}\right)^{\frac{1}{1+\beta}} \right\}.$$

Then for any $z \in Q_{\theta R}, r \in [\theta(1-\theta)R, (1-\theta)R]$, denote $\Psi(z, r) \coloneqq \varphi(z, r, v, p) + \psi(z, r, f)$, we attempt to show

$$\Psi(z, \theta^k r) \leq \theta^{(1+\alpha)k}\epsilon_2, \forall k \geq 0, \tag{5.16}$$

which means for any $\rho \in (0, (1-\theta)R](\rho = \theta^k r)$,

$$\varphi(z, \rho, v) \leq \Psi(z, \rho) \leq \left(\frac{\rho}{r}\right)^{1+\alpha} \epsilon_2 \leq \left(\frac{\rho}{\theta r_0}\right)^{1+\alpha} \epsilon_2 \lesssim \rho^{1+\alpha}.$$

That is say, $v \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q_{\theta R}})$ by Campanato characterization. Then the remained work is to verify (5.16). Indeed, the condition implies $k = 0$ case:

$$\phi(z, r) \leq 1; \textbf{?}$$
$$\Psi(z, r) = \varphi(z, r, v, p) + \psi(z, r, f) \leq c\psi(z, r, v, p) + \psi(z, r, f)$$
$$\leq c\,(R/r)^{\frac{4}{3}}\,\psi(R, v, p) + (r/r_0)^{1+\beta}\psi(z, r_0, f)$$
$$\leq c_\theta\epsilon_1 + \frac{\epsilon_2}{2} \leq \epsilon_2$$

for ϵ_1 small enough. Moreover, the iteration follows by the lemma:

$$\begin{aligned}
\phi(z,\theta^k r) =&1;\\
\Psi(z,\theta^k r) =&\varphi(z,\theta^k r,v,p)+\psi(z,\theta^k r,f)\\
\leq& C\theta^{1+\frac{2}{3}}\left(\varphi(z,\theta^{k-1}r,v,p)+\psi(z,\theta^{k-1}r,f)\right)+\theta^{1+\beta}\psi(z,\theta^{k-1}r,f)\\
\leq&\theta^{1+\alpha}\Psi(z,\theta^{k-1}r)\leq\cdots\leq\theta^{(1+\alpha)k}\epsilon_2.
\end{aligned}$$

Thus the claim finished. □

Theorem 5.27. *For any $s\in[1,\infty]$, there exists $\epsilon>0$, such that any (v,p,f) a suitable weak solution of* (NS) *in Ω_T, v is regular at $z\in\Omega_T$ if one of following conditions holds:*

1. $\dfrac{3}{q}+\dfrac{2}{s}\in[1,2], \liminf\limits_{r\to0} r^{-(\frac{3}{q}+\frac{2}{s}-1)}\|v-(v)_{B_r}\|_{L^sL^q(Q(z,r))}\leq\epsilon;$
2. $\dfrac{3}{q}+\dfrac{2}{s}\in[2,3], \liminf\limits_{r\to0} r^{-(\frac{3}{q}+\frac{2}{s}-2)}\|\nabla v\|_{L^sL^q(Q(z,r))}\leq\epsilon;$

Proof. It is convenient to denote homogenous quantities:

$$\begin{aligned}
&A(r)=\frac{1}{r}\|v\|^2_{L^\infty L^2(Q_r)}, B(r)=\frac{1}{r}\|\nabla v\|^2_{L^2L^2(Q_r)};\\
&C(r)=\frac{1}{r^2}\|v\|^3_{L^3(Q_r)}, \tilde{C}(r)=\frac{1}{r^2}\|v-(v)_{B_r}\|^3_{L^3(Q_r)}, D(r)=\frac{1}{r^2}\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_r)};\\
&G_1(r)=\frac{1}{r}\|v-(v)_{B_r}\|_{L^sL^q}(Q_r), G_2(r)=\frac{1}{r}\|\nabla v\|_{L^sL^q(Q_r)}.
\end{aligned}$$

For above quantities, there are rough dominations for $k\geq1$,

$$A/B/C/\tilde{C}/D/G(r)\leq c_k A/B/C/\tilde{C}/D/G(kr).$$

To process further, we'd like to show the following estimates for iteration: $\forall r\leq\frac{\rho}{2}$,

$$\begin{aligned}
&C(r)\lesssim\left(\frac{r}{\rho}\right)C(\rho)+\left(\frac{\rho}{r}\right)^2\tilde{C}(\rho), D(r)\lesssim\left(\frac{r}{\rho}\right)D(\rho)+\left(\frac{\rho}{r}\right)^2\tilde{C}(\rho);\\
&\tilde{C}(r)\lesssim A^{\frac{1}{s}}(r)E^{1-\frac{1}{s}}(r)G(r), (A+E)(r)\lesssim1+(C+D)(2r).
\end{aligned}\tag{5.17}$$

And all above estimates give out: $\forall r\leq\dfrac{\rho}{4}$,

$$\begin{aligned}
\tilde{C}\left(\frac{\rho}{2}\right)\lesssim&A^{\frac{1}{s}}\left(\frac{\rho}{2}\right)E^{1-\frac{1}{s}}\left(\frac{\rho}{2}\right)G\left(\frac{\rho}{2}\right)\lesssim\left(\frac{1}{s}A\left(\frac{\rho}{2}\right)+\left(1-\frac{1}{s}\right)E\left(\frac{\rho}{2}\right)\right)G\left(\frac{\rho}{2}\right)\\
\lesssim&(A+E)\left(\frac{\rho}{2}\right)G\left(\frac{\rho}{2}\right)\lesssim(1+(C+D)(\rho))G(\rho);\\
(C+D)(r)\lesssim&\left(\frac{r}{\rho}\right)(C+D)\left(\frac{\rho}{2}\right)+\left(\frac{\rho}{r}\right)^2\tilde{C}\left(\frac{\rho}{2}\right);\\
\lesssim&\left(\frac{r}{\rho}\right)(C+D)(\rho)+\left(\frac{\rho}{r}\right)^2(1+C(\rho)+D(\rho))\,G(\rho).
\end{aligned}$$

Set the constant as $c > 0$, then we choose $\theta \in (0, 1/4)$ so that $c\theta < 1/4$. By assumption, there is $r_0 > 0, \forall r \le r_0, G(r) < \dfrac{\theta^2 \epsilon_1}{1+8c}$. Then the estimate indicates:

$$\begin{aligned}
(C+D)(\theta r) \le& \frac{1}{2}(C+D)(r) + \theta^{-2}\left(1 + (C+D)(r)\right) G(r) \\
\le& \frac{1}{2}(C+D)(r) + \frac{\epsilon_1}{4} \textbf{?}, \\
\Longrightarrow (C+D)(\theta^k r) \le& \frac{1}{2^k}(C+D)(r) + \frac{\epsilon_1}{2}, \forall r < r_0.
\end{aligned}$$

Then for k big enough(depends on r), we have $(C+D)(\theta^k r) \le \epsilon_1$. Let $R = \theta^k r$, then the above criterion shows v is regular near 0. Following are the verification of (5.17):

1. $C(r) \lesssim \left(\frac{r}{\rho}\right) C(\rho) + \left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho)$:

$$\begin{aligned}
C(r) \lesssim& \frac{1}{r^2}\int_{Q_r} |v - (v)_{B_\rho}|^3 + \frac{1}{r^2}\int_{Q_r} |(v)_{B_\rho}|^3 \\
\lesssim& \left(\frac{\rho}{r}\right)^2 \left(\frac{1}{\rho^2}\int_{Q_\rho} |v - (v)_{B_\rho}|^3\right) + \frac{r}{\rho}\left(\frac{1}{\rho^2}\int_{Q_\rho} |v|^3\right) {}^{1} \\
\lesssim& \left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho) + \frac{r}{\rho} C(\rho);
\end{aligned}$$

2. $D(r) \lesssim \left(\frac{r}{\rho}\right) D(\rho) + \left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho)$: Since $\Delta p = \partial_i\partial_j(v^i v^j)$, then we decompose $p = \tilde{p} + h$, where $\Delta\tilde{p} = \zeta(\partial_i\partial_j(v^i v^j)), \zeta$ cutoff on $B_{\rho/2}$. Then

$$\|\tilde{p}\|_{L^{\frac{3}{2}}(B_\rho)} \lesssim \left\|\Gamma * (\partial_i\partial_j(v^i - (v)^i_{B_\rho})(v^j - (v)^j_{B_\rho}))\right\| \lesssim \left\|v - (v)_{B_\rho}\right\|_{L^3(B_\rho)},$$

and since $|h|^{\frac{3}{2}}$ is sub-harmonic[2], it comes

$$\begin{aligned}
D(r) \lesssim& \frac{1}{r^2}\int_{Q_r} |\tilde{p}|^{\frac{3}{2}} + \frac{1}{r^2}\int_{Q_r} |h|^{\frac{3}{2}} \lesssim \frac{1}{r^2}\int_{Q_\rho} |\tilde{p}|^{\frac{3}{2}} + \frac{r}{\rho^3}\int_{B_\rho} |h|^{\frac{3}{2}} \\
\lesssim& \left(\frac{r}{\rho}\right)\left(\frac{1}{\rho^2}\int_{B_\rho} |p|^{\frac{3}{2}}\right) + \left(\frac{r}{\rho^3} + \frac{1}{r^2}\right)\left(\int_{B_\rho} |\tilde{p}|^{\frac{3}{2}}\right) \\
\lesssim& \left(\frac{r}{\rho}\right) D(\rho) + \left(\frac{\rho}{r}\right)^2 \tilde{C}(\rho).
\end{aligned}$$

[1]Here $\int_{Q_r} |(v)_{B_\rho}|^3 = \int_{-r^2}^0 |B_r||(v)_{B_\rho}| \le (\frac{r}{\rho})^3 \int_{-\rho^2}^0 |B_\rho||(v)_{B_\rho}| \le (\frac{r}{\rho})^3 \int_{B_\rho} |v|^3$.

[2]Sub-harmonicity is preserved under convex function.

3. $\tilde{C}(r) \lesssim A^{\frac{1}{s}}(r)E^{1-\frac{1}{s}}(r)G(r)$: Apply Holder interpolation:

$$\begin{aligned}\tilde{C}(r) =& \frac{1}{r^2}\|v-(v)_{B_r}\|^3_{L^3(Q_r)} \lesssim \frac{1}{r^2}\|v-(v)_{B_r}\|^{\frac{2}{3}}_{L^\infty L^2}\|v-(v)_{B_r}\|^{2-\frac{2}{s}}_{L^2L^6}\|v-(v)_r\|_{L^sL^q}\\ \lesssim& \left(\frac{1}{r}\|v\|^2_{L^\infty L^2}\right)^{\frac{1}{s}}\left(\frac{1}{r}\|\nabla v\|^2_{L^2L^2}\right)^{1-\frac{1}{s}}\|v-(v)_{B_r}\|_{L^sL^q} \simeq A^{\frac{1}{s}}(r)E^{1-\frac{1}{s}}(r)G(r);\end{aligned}$$

The second condition is similar: For $(q,s)=(3,1)$,

$$\begin{aligned}\|v-(v)_{B_r}\|^3_{L^3(B_r)} \lesssim& \|v-(v)_{B_r}\|^2_{L^3(B_r)}\|\nabla v\|_{L^3(B_r)} + r^{-\frac{3}{2}}\|v-(v)_{B_r}\|^3_{L^2(B_r)}\\ \leq& \|v\|^2_{L^2(B_r)}\|\nabla v\|_{L^3(B_r)}.\end{aligned}$$

Integrating in time and apply Holder interpolation :

$$\begin{aligned}\tilde{C}(r) =& r^{-2}\left\|\|v-(v)_{B_r}\|^3_{L^3(B_r)}\right\|_{L^1(-r^2,0)}\\ \lesssim& r^{-2}\left\|\|v\|^2_{L^2(B_r)}\|\nabla v\|_{L^3(B_r)}\right\|_{L^1(-r^2,0)}\\ \lesssim& r^{-2}\|v\|^2_{L^\infty L^2(Q_r)}\|\nabla v\|_{L^1L^3(Q_r)} = A(r)G_2(r).\end{aligned}$$

4. $(A+E)(r) \lesssim 1 + (C+D)(2r)$: We take $\phi \in \mathcal{D}(Q_{2r})$ cutoff on Q_r, then the local energy inequality gives:

$$\begin{aligned}\|v\|^2_{L^\infty L^2(Q_r)} + \|\nabla v\|^2_{L^2(Q_r)} \leq& \|v_0\|_{L^2(B_r)} + \int_{Q_{2r}} |v|^2(\partial_t\phi + \Delta\phi) + (|v|^2+2p)|v|\nabla\phi\\ \lesssim& 1 + r^{-2}\|v\|^2_{L^2(Q_{2r})} + r^{-2}\|v\|^3_{L^3(Q_{2r})} + r^{-1}\|vp\|_{L^1(Q_{2r})}.\end{aligned}$$

Take r^{-1} at both hand, we get that

$$\begin{aligned}A(r)+B(r) \lesssim& 1 + (r^{-2}\|v\|^3_{L^3(Q_{2r})})^{\frac{3}{2}} + r^{-2}\|v\|^3_{L^3(Q_{2r})}{}^{1}\\ &+ \left(r^{-2}\|v\|^3_{L^3(Q_{2r})}\right)^{\frac{1}{3}}\left(r^{-2}\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_{2r})}\right)^{\frac{2}{3}}\\ \lesssim& 1 + C(2r) + D(2r).\end{aligned}$$

□

[1] Notice $\|v\|_{L^2(Q_{2r})} \leq |Q|^{\frac{1}{2}-\frac{1}{3}}\|v\|_{L^3(Q_{2r})} \lesssim r^{-\frac{5}{6}}\|v\|_{L^3(Q_{3r})}$.

5.8.3 Singularity

Suppose v is a weak solution (NS) in Ω_T, we say v is regular at $z=(x,t)\in\overline{\Omega_T}$, if $v\in L^\infty(Q_{z,r}\cap\Omega_T)$ for some $r>0$, otherwise it is singular. The set of singular points is denote as

$$S=\{z\in\bar{\Omega}_T|v \text{ is singular at } z\};$$

and the projection to time $\Sigma=P_t(S)=\{t\in(0,T]|(x,t)\in S \text{ for some } x\in\bar{\Omega}\}$ is called the set of singular times. To measure singular times more precisely, we introduce the Hausdorff measure of $E\subset\mathbb{R}^n$:

$$H^\alpha(E)=\liminf_{\delta\to 0+}\left\{\sum_i r_j^\alpha \middle| E\subset\bigcup B(x_j,r_j)\right\}.$$

Clearly it is equivalent to Lebesgue measure when $\alpha=n$, and $H^\alpha(E)\lesssim m(E)$ if $\alpha\le n$.

Theorem 5.28. *Let v be a Leray-Hopf weak solution in Ω_T with zero force, satisfies the strong energy inequality. Then $H^{\frac{1}{2}}(\Sigma)=0$.*

Proof. Denote following time sets

$$\begin{aligned}&\Sigma_1=\{t\in(0,T)|\|v\|_{L^6}=\infty\};\\&\Sigma_2=\{t\in[0,T)|\text{ strong energy inequality does not launch at } t\ \};\\&U_1=(0,T)\setminus\Sigma_1, U_2=[0,T)\setminus\Sigma_2.\end{aligned}$$

Clearly Σ_1,Σ_2 are Lebesgue zero-measurable by the assumption. For any $t\in U_2, v(t)\in L^2_\sigma\cap L^6_\sigma$, then it generate a mild LHWS v'(regular of course) at time interval:

$$[t,t+T(t)),T(t)\gtrsim\|v\|_6^{-4}.$$

For $t\in U_1$, v is a LHWS at interval $[t,T)$. Consequently, the strong-weak uniqueness implies $v'=v$ at interval $I(t)=(t,\min\{T(t),T\})$, if $t\in U_1\cap U_2$. We set

$$U_3=\bigcup_{t\in U_1\cap U_2} I(t),\Sigma_3=(0,T)\setminus U_3.$$

Then v must be regular in U_3, and the singular times set $\Sigma\subset\Sigma_3\cup\{T\}$. Now it is enough to check $H^1(\Sigma_3)=0$. To apply Vatali Covering lemma, we set $\Sigma_3^\delta=\Sigma_3\cap[\delta,\min\left\{T-\delta,\frac{1}{\delta}\right\}]$. Since $|\Sigma_3|=0$, we can choose V neighbor Σ_3^δ with $\|v\|_{L^2L^6(\Omega\times V)}\le\epsilon$. Now for any $t\in\Sigma_3^\delta$,

$$\exists r_t\le\delta, B(t,r_t)\subset V\Longrightarrow\Sigma_3^\delta\subset\bigcup_{t\in\Sigma_3^\delta}B(t,r_t)\subset V\Longrightarrow\exists B(t_j,r_j)\text{ disjoint},\Sigma_3^\delta\subset\bigcup_j B(t_j,5r_j).$$

Notice for any $t\in(t_j-r_j,t_j),t\in U_3,t_j\in\Sigma_3^\delta$, the blow-up rate is $\|v(t)\|_6\gtrsim\|t_j-t\|^{-\frac{1}{4}}$. Then

$$\int_{t_j-r_j}^{t_j}\|v(t)\|_6^2\gtrsim^{1}\int_{t_j-r_j}^{t_j}(t_j-t)^{-\frac{1}{2}}\gtrsim r_j^{\frac{1}{2}}\Longrightarrow H^{\frac{1}{2}}(\Sigma_3^\delta)\le\sum r_j^{\frac{1}{2}}\lesssim\|v\|_{L^2L^6(\Omega\times V)}\le\epsilon.$$

As δ,ϵ varies arbitrarily, we have $H^{\frac{1}{2}}(\Sigma)\leq H^{\frac{1}{2}}(\Sigma_3)=0$. □

To measure the singular points, a parabolic version Hausdorff measure is need:

$$P^\alpha(E)=\liminf_{\delta\to 0+}\left\{\sum_i r_j^\alpha \,\middle|\, E\subset\bigcup Q(x_j,r_j)\right\}.$$

Theorem 5.29. *Let (v,p) be a suitable weak solution of (NS) in Ω with force $f\in L^{2,2\lambda+1}$, $\lambda\in(0,2)$. Then $P^1(S)=0$.*

Proof. The idea is similar with former one, where we use $\epsilon-$criterion to determine the blow-up rate. To apply parabolic Vatali covering, it is proper to define a shifted cylinder:

$$Q^*(z,r)=B(x,r)\times(t-\frac{7}{8}r^2,t+\frac{1}{8}r^2),$$

so that $Q^*(z,r)\ni z$ and $Q^*(z,r)\supset Q(z,\frac{1}{2}r)$. For the set S of singular points, we cut it off by $R\Subset\Omega\times(0,T]$ so that $S\cap R\Subset\Omega\times(0,T+1)$. Fix $\delta>0,\epsilon>0,\exists V\overset{\circ}{\supset}S\cap R$, $\int_V\|\nabla v(t)\|^2<\epsilon$. And for any $z\in S\cap R$, we can find $r_z<\delta/5$ small enough s.t.

$$Q^*(z,r_z)\subset V,\int_{Q(z,r_z/2)}|\nabla v|^2>cr_z.$$

Consequently, we get

$$S\cap R\subset\bigcup Q^*(z,r_z)\Longrightarrow\exists Q^*(z_j,r_j)\text{ disjoint},S\cap R\subset\bigcup Q^*(z_j,5r_j)$$

by parabolic Vatali covering. And thus

$$\sum_j r_j\leq C\sum_j\int_{Q(z_j,r_j/2)}|\nabla v|^2\leq C\sum_j\int_{Q^*(z_j,r_j)}|\nabla v|^2\leq C\int_V|\nabla v|^2\leq C\epsilon.$$

Since R,δ,ϵ vary arbitrarily, we see $P^1(S)=0$. □

5.9 Self-Similar Solution

By the scaling relation, we seek for a solution (v,p,f) satisfies $(v,p,f)=(v^\lambda,p^\lambda,f^\lambda),\forall\lambda>0$ given a conic domain Ω[1]. Clearly the solution is determined by its value at time ±1, as

$$\begin{aligned}(v,p,f)(x,t)=&((t^{-\frac{1}{2}}\sqrt{t}v,t^{-1}(\sqrt{t})^2p,t^{-\frac{3}{2}}(\sqrt{t})^3f)(\sqrt{t}\frac{x}{\sqrt{t}},(\sqrt{t})^2\cdot 1)\\=&(t^{-\frac{1}{2}}v^{\sqrt{t}},t^{-1}p^{\sqrt{t}},t^{-\frac{3}{2}}f^{\sqrt{t}})(\frac{x}{\sqrt{t}},1)\\=&(t^{-\frac{1}{2}}v,t^{-1}p,t^{-\frac{3}{2}}f)(\frac{x}{\sqrt{t}},1).\end{aligned}$$

[1]Notice that $\|v(t)\|_6\gtrsim(t_j-t)^{\frac{1}{4}}$ holds for almost every t by the definition of U_3.

[1]Especially, the half/whole space.