Spectral charaterization of self-adjointness.

Here we'd like to fill the gap between symmetricity and self-adjointness by the spectral charaterization.

> **Theorem.** Suppose $A: D(A) \subseteq H \to H$ is closed and densed defined. Then A is self-adjoint if and only if A is symmetric and $\sigma(A) \subseteq \mathbb{R}$.

Remark. Indeed we can remove the symmetric condition: the equivalence between self-adjointness and real spectrum hold for a much larger class, called normal operators. But the proof for such general case need the application of spectral theorem. So we'd to prove the symmetric version here and apply it to construct the self-adjoint extension.

Proof: $\Rightarrow$ Suppose A is self-adjoint. then it's symmetric naturally, following we show that $(\lambda - A)$ is bijective (and thus invertible, since closed + everywhere defined), for every $\lambda \in \mathbb{R}^c$.

Injection: It is enough to establish a lower bound for $v = (\lambda - A)u \in D(A)$, we just notice the image part of the following identity:

$$\langle v, u \rangle = \lambda \langle u, u \rangle - \langle Au, u \rangle. \quad \langle Au, u \rangle = \langle u, Au \rangle = \overline{\langle Au, u \rangle}$$

The symmetricity ensure the reality of $\langle Au, u \rangle$, and $\lambda \in \mathbb{R}^c$ ensure $\operatorname{Im} \lambda \neq 0$, The we see $|\operatorname{Im}\lambda|\, \|u\|^2 \leq \|u\|\,\|v\| \Rightarrow \|u\| \leq \frac{\|v\|}{|\operatorname{Im}\lambda|}$.

Subjection: We first show the closedness of $R(\lambda-A)$. We choose $\{\overset{v_n=:}{\overbrace{(\lambda-A)u_n}}\} \subseteq R(\lambda-A)$ a Cauchy sequence. The the above lower bound ensure $\{u_n\}$ is also Cauchy and $u_n \to u_0 \in H$. We show that $u_0 \in D(A)$ and $v_n = (\lambda-A)u_n \to (\lambda-A)u_0 \in D(A)$.

The idea is to use the closedness of A: we notice $Au_n = \lambda u_n + v_n$ is Cauchy and so does (u_n, Au_n). $\overset{A \text{ closed}}{\Longrightarrow} u_0 \in D(A)$, $(u_n, Au_n) \to (u_0, Au_0)$, which finish the closedness of $R(\lambda-A)$.

Besides, if $\overline{R(\lambda-A)} \neq H$, then there exists $0 \neq w \in H$ such that

$$\langle (\lambda-A)u, w\rangle = 0, \quad \forall u \in D(A).$$

$$\Rightarrow \langle u, \bar{\lambda} w\rangle = \langle \lambda u, w\rangle = \langle Au, w\rangle = \langle u, Aw\rangle, \quad \forall u \in D(A).$$

$$\Rightarrow \quad Aw = \bar{\lambda} w \Rightarrow \langle Aw, w\rangle = \bar{\lambda}\langle w, w\rangle.$$

This implies $\langle Aw, w\rangle$ is not real since $w \neq 0$ and $\lambda \in \mathbb{R}^c$, which contradicts to the symmetric assumption.

In conclusion, the $\Rightarrow$ side is done.

As for $\Leftarrow$ side, we need to show $\mathbb{R}^c \subseteq \rho(A)$ implies A is self-adjoint. Indeed, this basic observation is that $\lambda \subseteq \rho(A) \Leftrightarrow \bar{\lambda} \subseteq \rho(A)$. Consequently,

$$\langle (\lambda-A)y, z\rangle = \langle y, (\bar{\lambda}-A)z\rangle, \quad \forall y, z \in D(A).$$

$$\Leftrightarrow \quad \langle x, (\bar{\lambda}-A)^{-1}s\rangle = \langle (\lambda-A)^{-1}x, s\rangle, \quad \forall x, s \in R(\lambda-A) = R(\bar{\lambda}-A) = H.$$

We use this identity to prove the self-adjointness: take $v \in D(A^*)$, $x \in H$

($\lambda \in \rho(A)$ implies $(\lambda-A)^{-1}x$ exists and $(\lambda-A)^{-1}x \in D(A)$).

$$\begin{aligned}\langle v, x\rangle &= \langle v, (\lambda-A)\underbrace{(\lambda-A)^{-1}x}^{y \in D(A)}\rangle \\ &= \langle (\bar{\lambda}-A^*)v, (\lambda-A)^{-1}x\rangle \\ &= \langle (\bar{\lambda}-A)^{-1}(\bar{\lambda}-A^*)v, x\rangle.\end{aligned}$$

This implies $v = (\bar{\lambda}-A)^{-1}(\bar{\lambda}-A^*)v \in D(\bar{\lambda}-A) = D(A)$. And so A is self-adjoint.

Now we step into the first task in this lecture. The Friedrichs self-adjoint extension: we will generically extend a general class of symmetric operators into the self-adjoint domain, which is called semi bounded.

Definition: $A: D(A) \subseteq H \to H$ is called semi-bounded if

$$\exists C \in \mathbb{R}, \text{ s.t. } \langle Au, u\rangle \geq c\langle u, u\rangle, \quad \forall u \in D(A).$$

We called c the lower-bound of A.

Proposition: A semi-bounded symmetric operator $A: D(A) \subseteq H \to H$ has a self-adjoint extension $\widehat{A}$ with the same bound.

Proof. We consider $c=1$ case first, the general case can be obtained by consider $A + (1-c)I$ and $(\widehat{A+(1-c)I}) + (c-1)I$ is self-adjoint with c-bound.

Now we construct the extension by define the following inner product:

$$\langle u, v\rangle_{D(A)} =: \langle u, Av\rangle_H, \quad \forall u, v \in D(A).$$

$\begin{cases} \text{symmetricity implies } \langle u, v\rangle = \overline{\langle v, u\rangle}. \\ \text{1-semibounded implies } \langle u, u\rangle \geq 0. \end{cases}$

We define V as the completion of $D(A)$ under this inner product, equipped with the same norm. Then $(V, \langle\cdot,\cdot\rangle_V) \hookrightarrow (H, \langle\cdot,\cdot\rangle_H)$ since

$$\langle u, u\rangle_V = \langle u, Au\rangle_H \geq \langle u, u\rangle_H.$$

Now we define the extended domain as:

$\langle w, \Delta u\rangle \leq C\|w\|_{L^2}\}$

i.e. $\langle\cdot, u\rangle_V$ is a bounded functional in $(V, \langle\cdot,\cdot\rangle_H)$

$$D(\widehat{A}) = \{u \in V \mid \exists v \in H, \text{ s.t. } \langle\cdot, u\rangle_V = \langle\cdot|_V, v\rangle_H\}$$

since V is dense in H).

and we attempt to define $\widehat{A}: u \mapsto v$. (well-defined since V is dense).

1. $\widehat{A}$ is bijective. Indeed, for any $v \in H$, we see $\langle\cdot|_V, v\rangle_H$ is a bounded functional in $(V, \langle\cdot,\cdot\rangle_V)$ since

$$|\langle w, v\rangle_H| \leq \|w\|_H \|v\|_H \leq \|w\|_V \|v\|_H, \quad \forall w \in V.$$

Consequently, the Riesz representation ensure the unique $u \in V$ such that: $\langle\cdot|_V, u\rangle_H = \langle\cdot, u\rangle_V$. which implies $\widehat{A}: u \mapsto v$ is injective and surjective.

1. Clearly $A \subseteq \widehat{A}$. Since for $u \in D(A)$, $\quad \widehat{A}: V \to H$.

$$\langle\cdot, u\rangle_V = \langle\cdot|_V, Au\rangle_H \Rightarrow \widehat{A}u = Au.$$

2. Self-adjointness. It is enough to show $\widehat{A}^{-1}$ is symmetric, then $\widehat{A}$ is surjective

implies that $(\breve{A})^{-1}$ is everywhere defined, which implies $\breve{A}^{-1}$ is self-adjoint and so does $\breve{A}$. Indeed, (thus $D(A^*) \subseteq D(A)$.)

$$\breve{A} \text{ is symmetric} \Rightarrow (\breve{A})^{-1} \text{ is symmetric} \xRightarrow{(\breve{A})^{-1} \text{ is e.d.}} (\breve{A})^{-1} \text{ is self-adjoint} \Rightarrow \breve{A} \text{ is self-adjoint.}$$

1) $\breve{A}$ is symmetric: $\forall w, u \in D(\breve{A})$, $\breve{A}$在H中自伴性 $\Leftrightarrow$ I在V中自伴.

$$\langle w, \breve{A}u \rangle_H = \langle w, u \rangle_V = \overline{\langle u, w \rangle_V} = \overline{\langle u, \breve{A}w \rangle_H} = \langle \breve{A}w, u \rangle.$$

2) $\breve{A}^{-1}$ is symmetric:

$$\langle \breve{A}^{-1}w, u \rangle_H = \langle \breve{A}^{-1}w, \breve{A}\breve{A}^{-1}u \rangle_H = \langle \breve{A}\breve{A}^{-1}w, \breve{A}^{-1}u \rangle$$
$$= \langle w, \breve{A}^{-1}u \rangle_H .$$

Self-adjoint extension of Laplacian:

$$-\Delta : D \subseteq L^2 \to L^2. \qquad \langle u, v \rangle_V = \langle (-\Delta + I)u, v \rangle_2 = \langle u, v \rangle + \langle \nabla u, \nabla v \rangle - \int_{\partial\Omega} \frac{\partial u}{\partial n}$$