Preliminary about spectral theorem.

Definition 1. Suppose A is closed and densely defined in H, then A is normal if $A^*A = A^*A$.

Definition 2. We define $E: \mathcal{B}(\mathbb{C}) \to P(H)$ a spectral measure, if

① $E(\mathbb{C}) = I$; ② $E(\sqcup_i \Omega_i) = \sum_i E(\Omega_i)$.

Spectral Theorem. Suppose A is a normal operator in H, then there is a unique spectral measure E, such that (E support on $\sigma(A)$).

$$A = \int_{\mathbb{C}} \lambda \, dE(\lambda) = \int_{\sigma(A)} \lambda \, dE(\lambda),$$

Moreover, for $f \in \mathcal{M}(\sigma(A))$, we can define $f(A) = \int f(\lambda) \, dE(\lambda)$.

Some direct application.

① Suppose A is a normal operator, then

1) A is self-adjoint if and only if $\sigma(A) \subseteq \mathbb{R}$;

2) A is non-negative if and only if $\sigma(A) \subseteq [0, \infty)$;

② Suppose A is self-adjoint, then we have for $\lambda_0 \in \rho(A)$,

$$\|(\lambda_0 - A)^{-1}\| = \operatorname{dist}(\lambda_0, \sigma(A))^{-1}.$$

Definition of Riesz projection: Suppose A is a normal operator in X and correspond to E, let Δ be a connected component of $\sigma(A)$, we define

$$P_\Delta(A) =: \int \chi_\Delta(\lambda) \, dE(\lambda) = \int \frac{1}{2\pi i} \int_\Gamma \frac{1}{z-\lambda} dz \, dE(\lambda).$$

$$= \frac{1}{2\pi i} \int_\Gamma \int \frac{1}{z-\lambda} dE(\lambda) \, dz = \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} dz.$$

$$\lambda \in \mathbb{C} \xrightarrow{(\lambda I - T)^{-1} \text{ exists?}} \begin{cases} N: \lambda \in \sigma_p(T), \\ Y: \xrightarrow[\overline{(D((\lambda I - T)^{-1})} = X?)]{\overline{R(\lambda I - T)} = X?} \begin{cases} N: \lambda \in \sigma_r(T), \\ Y: \xrightarrow{(\lambda I - T)^{-1} \text{ is bounded?}} \begin{cases} N: \lambda \in \sigma_c(T), \\ Y: \lambda \in \rho(T). \end{cases} \end{cases} \end{cases}$$

Consequently, we have $\mathbb{C} = \rho(T) \sqcup \rho(T) =: \rho(T) \sqcup \sigma_c(T) \sqcup \sigma_r(T) \sqcup \sigma_p(T)$, last three called ***continuous spectrum***, ***residual spectrum***, ***point spectrum*** respectively.

singular discrete

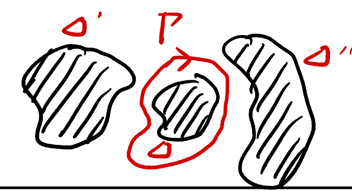
Lecture 4. The essential and discrete spectrum.

In this lecture, we'd like to decompose the spectrum into two different types by the Riesz projection, and we establish some important criterion for such classification and apply them to find the exact spectrum of Laplacian operator.

Definition 1: Let $A: D(A) \subseteq X \to X$ be closed when X is Banach. Suppose Δ is a connected component of $\sigma(A)$. now we define the Riesz projection of A at Δ as

$$P_\Delta =: \frac{1}{2\pi i}\int_\Gamma (z-A)^{-1}dz$$

where Γ is a admissible contour in $\rho(A)$ round only Δ.



Remark: Particularly, if $\Delta = \{\lambda\}$, we'd like to denote $P_\Delta = P_\lambda$. (For example, the spectrum for the compact operator).

We'd like to list the basic properties of the Riesz projection:

Proposition 1. Suppose $A: D(A) \subseteq X \to X$ is closed when X is Banach, $\Delta \subseteq \sigma(A)$ is a connected component. Then

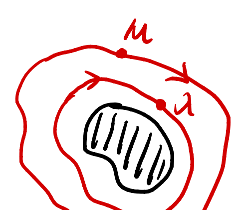
Sigal lemma 6.1.

① The Definition of P_Δ is free with choice of Γ. (well-definedness).

② P_Δ is a projection into $\ker(\Delta - A)$. ($P_\Delta^2 = P_\Delta$, $\underbrace{P_\Delta X}_{GM} \supseteq \underbrace{\ker(\Delta - A)}_{AM}$).

③ if A is self-adjoint, then P_Δ is a orthnormal projection and $P_\Delta X = \ker(\Delta - A)$.

Proof: ② Just take another contour Γ' and



$$P_\Delta \cdot P_\Delta = \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1} d\lambda \cdot \frac{1}{2\pi i}\int_{\Gamma'} (\mu - A)^{-1} d\mu$$

↓ resolvent formula

$$= \frac{1}{(2\pi i)^2}\int_{\Gamma'}\int_\Gamma \frac{1}{\mu - \lambda}\left((\lambda - A)^{-1} - (\mu - A)^{-1}\right) d\lambda\, d\mu$$

$$= \frac{1}{2\pi i}\int_\Gamma \underbrace{\left(\frac{1}{2\pi i}\int_{\Gamma'} \frac{1}{\mu - \lambda} d\mu\right)}_{=1} (\lambda - A^{-1})\, d\lambda$$

residue theorem.

$$+ \frac{1}{2\pi i}\int_{\Gamma'} \underbrace{\left(\frac{1}{2\pi i}\int_\Gamma \frac{1}{\mu - \lambda} d\lambda\right)}_{=0,} (\mu - A)^{-1} d\mu = P_\Delta.$$

Besides, for $x \in \ker(\Delta - A)$, i.e. $\exists \lambda_0 \in \Delta$, $Ax = \lambda_0 x$

$$\Rightarrow (\lambda - A)^{-1} x = (\lambda - \lambda_0)^{-1} x, \ \forall \lambda \in P \subseteq \rho(A),$$

$$\Rightarrow P_\Delta x = \frac{1}{2\pi i} \int_P (\lambda - A)^{-1} x \, d\lambda = \frac{1}{2\pi i} \int_P (\lambda - \lambda_0)^{-1} x \, d\lambda = x.$$

This implies $P_\Delta X \supseteq \ker(\Delta - A)$.

③ Here I only find the proof for isolated point set $\Delta = \{\lambda_0\}$. In this case, we attempt to show that:

1) orthonormality: $P_{\lambda_0} = P_{\lambda_0}^*$;

2) $P_{\lambda_0} X \subseteq \ker(\lambda_0 - A)$, i.e. $(\lambda_0 - A) P_{\lambda_0} = 0$.

Proof of 1): we just choose P: $|\lambda - \lambda_0| = r$, and take $\lambda = \lambda_0 + re^{i\theta}$, then we find:

$$P_{\lambda_0} = \frac{1}{2\pi i} \int_P (\lambda - A)^{-1} d\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_A(\lambda_0 + re^{i\theta}) \, re^{i\theta} \, d\theta$$

Proof of 2): $$(\lambda_0 - A) P_{\lambda_0} = \frac{1}{2\pi i} \int_P (\lambda_0 - A)(\lambda - A)^{-1} d\lambda$$

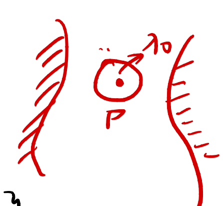
$$= \frac{1}{2\pi i} \int_P (\lambda_0 - \lambda)(\lambda - A)^{-1} d\lambda.$$

It is enough to show $(\lambda_0 - \lambda)(\lambda - A)^{-1}$ is analytic inside P and then Cauchy's theorem implies the integration vanishes. This is due to the resolvent estimate:

$$\|(\lambda - A)^{-1}\| \leq \operatorname{dist}(\lambda, \sigma(A))^{-1}.$$ → this may fail for Δ not islated.

Once we choose that P is small enough around λ_0 such that

$$|\lambda_0 - \lambda| = \operatorname{dist}(\lambda, \sigma(A)), \ \forall \lambda \in \rho(A).$$



Then we have: $(\lambda_0 - \lambda)(\lambda - A)^{-1}$ is uniformly bounded inside $U / \{\lambda_0\}$, which implies λ_0 is a pickable singularity and we can extend naturally, which finish the proof.

Remark. For a isolated spectrum point λ_0 we say $\dim P_{\lambda_0} X$ is the algebraic multiplicity of λ_0, and $\dim \ker(\lambda_0 - A)$ is the geometric multiplicity of λ_0.

The above theorem state the following fact: The AM $\geqslant$ GM for any isolated spectrum point of a closed operator. Moreover, if self-adjoint, they coincide exactly.

Essential spectrum and the discrete spectrum.

Now we can classify the spectrum of a closed operator into following two kinds:

1) discrete spectrum $\sigma_d(A)$: $\lambda \in \sigma_d(A)$ if λ is isolated + P_λ finite rank.

2) essential spectrum: $\sigma_{ess}(A)$: $\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A)$.

(there are some equivalent definition for $\sigma_{ess}(A)$ by Fredholm operators.)

Our following main goal is to establish a criterion for essential spectrum and apply it to figure out the spectrum of Laplacian operator on $\mathbb{R}^d$.

We state the main result here:

Theorem. (Weyl). Suppose $A: D(A) \subseteq X \to X$ is self-adjoint, then $\lambda \in \sigma_{ess}(A)$ if and only if there is a Weyl sequence $\{u_n\}$ for λ, i.e.

1) $\|u_n\| = 1$, 2) $u_n \rightharpoonup 0$; 3) $(\lambda - A) u_n \to 0$.

Remark: This is the criterion for the whole spectrum if we remove 2).

The proof is a complete argument and we'd like to sketch it here.

For self-adjoint operator:

$$\lambda \in \sigma_d(A) \iff \lambda \text{ is isolated } + \dim P_\lambda X < \infty$$

$\Updownarrow$ self-adjoint $\Updownarrow$

$$\lambda \in \rho(A|_{\ker(\lambda-A)^\perp}) + \dim \ker(\lambda - A) < \infty$$

$$\lambda \in \sigma_{ess}(A) \iff \lambda \in \sigma(A|_{\ker(\lambda-A)^\perp}) \text{ or } \dim \ker(\lambda - A) = \infty.$$

(since $\lambda - A|_{\ker(\lambda-A)^\perp}$ always has inverse, then the only case is the inverse is unbounded).

Proof of the theorem: $\Rightarrow$ suppose $\lambda \in \sigma_{ess}(A)$, then:

① if $\dim \ker(\lambda - A) = \infty$, just take orthonormal functions $\{u_n\} \subseteq \ker(\lambda - A)$.

we have clearly $\|u_n\| = 1$ and $(\lambda - A) u_n = 0$, besides $u_n \rightharpoonup 0$ since

$V_\infty = \text{span}\{u_1, \cdots, u_n, \cdots\}$ is dense in X.

② if $\dim \ker(\lambda - A) < \infty$ and $(\lambda - A_1)^{-1}$ is unbounded,

we can choose a sequence

$$\|v_n\| = 1, \quad \|(\lambda - A_1)^{-1} v_n\| \to \infty,$$

let $u_n = \dfrac{(\lambda - A_1)^{-1} v_n}{\|(\lambda - A_1)^{-1} v_n\|}$ $\in \ker(\lambda - A)^\perp$ exactly, then $\|u_n\| = 1$ and $\|(\lambda - A_1) u_n\| \to 0$.

As for the weak convergence, it is enough to prove:

$$\langle u_n, f\rangle \to 0 \quad \forall f \in \ker(\lambda - A)^\perp,$$

If we can establish the following density:

Claim: $D(((\lambda - A_1)^{-1})^*)$ is dense in $X_1 = \ker(\lambda - A)^\perp$.

(use the criterion $R(A \pm i) = H$)

Proof: $\lambda - A_1 : D \subseteq X_1 \to X_1$ has dense range and self-adjoint

$\Rightarrow (\lambda - A_1)^{-1}$ exists and self-adjoint.

$\Rightarrow ((\lambda - A_1)^{-1})^*$ is densely defined.

consequently, we see for $f \in D(((\lambda - A_1)^{-1})^*)$

$$\langle u_n, f\rangle = \frac{\langle (\lambda - A_1)^{-1} v_n, f\rangle}{\|(\lambda - A_1)^{-1} v_n\|} = \frac{\langle v_n, (\lambda - A_1)^{-1} f\rangle}{\|(\lambda - A_1)^{-1} v_n\|} \to 0.$$

And thus we finish the necessity.

$\Leftarrow$: Suppose $\{u_n\}$ is the Weyl's sequence and WLOG $\dim \ker(\lambda - A) < \infty$,

we attempt to show that $(\lambda - A_1)$ has a unbounded inverse.

Indeed, let $\{\Phi_i\}$ be the finite orthonormal basis for $\ker(\lambda - A)$, then:

$$\|P_\lambda u_n\|^2 = \sum \langle u_n, \Phi_i\rangle \to 0 \text{ since } u_n \rightharpoonup 0.$$

This implies: $\|(1-P_\lambda)u_n\| \to 1$. Take $v_n = (1-P_\lambda)u_n$,

we have: $\begin{cases} \|v_n\| \to 0, \\ (\lambda - A_1)v_n = (\lambda - A_1)u_n \to 0. \end{cases}$ $(\lambda - A_1): X_1 \to X_1.$

And so $(\lambda - A_1)^{-1}$ is unbounded.

Direct deomposition of the closed operator.

Suppose A is closed on X, P_1, P_2 decompose $\begin{cases} \sigma(A) = \sigma_1 + \sigma_2. \\ X = X_1 + X_2 = P_1 X + P_2 X, \end{cases}$

then we have: $A = A_1 \oplus A_2$, $\begin{cases} A_1 \text{ invariant on } X_1, \ \sigma(A_1) = \sigma_1, \\ A_2 \text{ invariant on } X_2, \ \sigma(A_2) = \sigma_2. \end{cases}$

Proof: consider for A_1 only, we show: $AP_1 = P_1 A$. (which implies $R(A_1) \subseteq X_1$).

$P_1 A x = \frac{1}{2\pi i}\int_P (\lambda - A)^{-1} A \, d\lambda = \frac{1}{2\pi i}\int_P A(\lambda - A)^{-1} d\lambda$

$= \frac{1}{2\pi i}\int_P (\lambda(\lambda-A)^{-1} - I)x \, d\lambda = A P_1 x$

Claim: λ is isolated $\Leftrightarrow \lambda \notin \sigma(A_2)$.

$\Rightarrow$ λ is islated $\Rightarrow$ $X_1 = P_\lambda X = \ker(\lambda - A)$

$X_2 = (1-P_\lambda)X = X_1^\perp = \ker(\lambda - A)^\perp$.

$\Rightarrow$ $A_2 = A|_{X_2}$ is invariant in X_2,

since $\lambda \in \sigma(A_1)$. ($A_1 = A|_{\ker(\lambda - A)}$), so $\lambda \bar{\in} \sigma(A_2)$

$\Leftarrow$ show $\lambda + z \in \rho(A)$ for small z $\Leftrightarrow$ $\lambda + z - A$ is invertible

$\Leftrightarrow$ $\lambda + z - A|_M$ $M = \ker(\lambda - A)$. are all invertible

$\lambda + z - A|_{M^\perp}$

since $\lambda \bar{\in} \sigma(A_2)$.

Spectrum of Laplace Operator

We discuss about the spectrum of Laplacian operator.

Theorem 4.24. *The spectrum of* $-\Delta : H^2 \subset L^2 \to L^2(\mathbb{R}^n)$ *is*

$$\sigma(-\Delta) = \sigma_{ess}(-\Delta) = [0, \infty).$$

Proof. As we already show that $-\Delta$ is self-adjoint, following we check that $-\Delta$ is positive so that $\sigma(-\Delta) \subset [0, \infty)$:

$$\langle u, -\Delta u\rangle_2 = \langle \hat{u}, \xi^2 \hat{u}\rangle_2 = \int \xi^2 \hat{u}^2 d\xi \geq 0.$$

It remains to show $\sigma_{ess}(-\Delta) \supset (0, \infty)$, so that $\sigma(-\Delta) = [0, \infty)$ as the spectrum is closed. And consequently $\sigma_{ess}(-\Delta) = \sigma(-\Delta) = [0, \infty)$ since 0 is not a isolated point.

Now for any $\lambda > 0$, we attempt to construct a sequence u_k by

$$\underline{\hat{u}_k(\xi) = (2\pi k)^{\frac{n}{2}} e^{-k^2(\xi-\xi_0)^2}, \xi_0^2 = \lambda}.$$ Gaussian Wave package.

and check that it is a Weyl's sequence for λ and $-\Delta$:

1. $\|u_k\|_2 = 1$. Indeed,
$$\|u_k\|^2 = \|u_k\|^2 = \langle \hat{u}_k, \hat{u}_k\rangle = (2\pi k)^n \int e^{-2k^2(\xi-\xi_0)^2} d\xi = (2\pi k)^n (2\pi k)^{-n} = 1.$$

2. $\|u_k\| \rightharpoonup 0$. Since $\mathcal{S}$ is dense in L^2, we can only test with $f \in \mathcal{S}$:
$$\begin{aligned} \langle u_k, f\rangle = \langle \hat{u}_k, \hat{f}\rangle =&(2\pi k)^{\frac{n}{2}} \int e^{-k^2(\xi-\xi_0)^2} \hat{f}(\xi) d\xi \\ (\eta = k(\xi - \xi_0)) =& \left(\frac{2\pi}{k}\right)^{\frac{n}{2}} \int e^{-\eta^2} \hat{f}\left(\frac{\eta}{k} + \xi_0\right) d\eta \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

3. $\|(\lambda - A)u_k\| \to 0$. Notice $u_k, \hat{u}_k \in \mathcal{S}(\mathbb{R}^n)$ by definition. Then
$$\begin{aligned} \|(\lambda + \Delta)u_k\|_2^2 =& \left\|(\lambda - \xi^2)\hat{u}_k\right\|_2^2 \\ =&(2\pi k)^n \int (\xi_0^2 - \xi^2) e^{-2k^2(\xi-\xi_0)^2} d\xi \\ =&(2\pi k)^n (\sqrt{2}k)^{-n} \int e^{-\eta^2} \left(\xi_0^2 - \left(\frac{\eta}{\sqrt{k}} + \xi_0\right)^2\right) d\eta \\ =& -(\sqrt{2}\pi)^n \int e^{-\eta} \left(\frac{\eta^2}{2k^2} + \sqrt{2}\frac{\eta}{k} \cdot \xi_0\right) d\eta, \end{aligned}$$
which varnish at rate k^{-1}.

In conlusion, u_k is a Weyl's sequence for any $\lambda > 0$. $\square$