Lecture 1.

Example: $A = \frac{d}{dt}: C^1(I) \subset C(I) \to C(I)$.

$u_n(t) = \sin nt$. $\quad \|u_n\|_{C(I)} = 1$.

$\|Au_n\|_{C(I)} = n \to \infty$.

$-\Delta: C_c^\infty(\Omega) \subseteq L^2 \to L^2$

$\searrow W^{k,p}$.

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1) Basic notions: Extension, dual / adjoint,

closedness, symmetricity, self-adjointness.

2) Spectral theory: functional calculus. $A = \int_{\sigma(A)} \lambda \, dE(\lambda)$

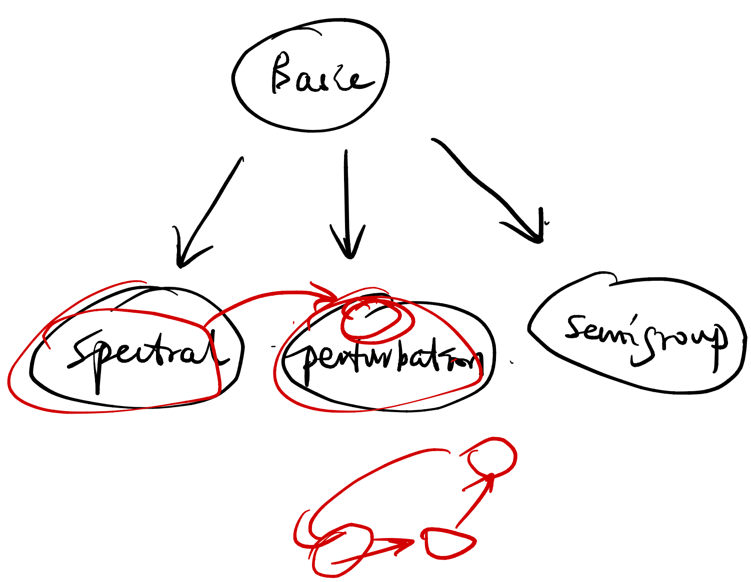
$A^k \leftarrow f(A) = \int_{\sigma(A)} f(\lambda)\, dE(\lambda)$.

e^{tA}, $\sqrt{A}$ $\qquad \downarrow$

$e^{t\Delta} \qquad e^{-t|\xi|^2} *$

3). $-\Delta \to -\Delta + V$. perturbation.

4). Semigroup.



$\partial_t u - Au = f$.

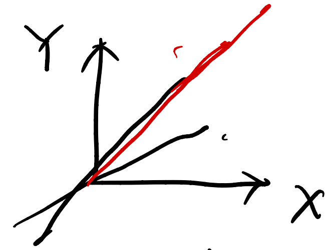
$\searrow \Delta$.

$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f \, ds$

$A \qquad\qquad A$.

Lecture 1. Unbounded operators. Yosida.

$A: D(A) \subseteq X \to Y$. Banach, X, Y. → locally convex



Definition.

1) extension: Graph of A: $\Gamma(A) = \{(x, Ax) \in X \times Y \mid x \in D(A)\}$.

$\tilde{A}$ is a extension of A, i.e. $\Gamma(\tilde{A}) \supseteq \Gamma(A)$.

i.e. $\forall x \in D(A)$, $x \in D(\tilde{A})$, $Ax = \tilde{A}x$.

2) dual / conjugate. dual of A: A':

$A: X \to Y$ is bounded, $A': Y' \to X'$, characterized:

△ counter. $(Ax, y') = (x, A'y')$, $\forall x \in X$, $y' \in Y'$.

3) $A: D(A) \subseteq X \to Y$ is densely defined. $\overline{D(A)} = X$.

(y', x_1')

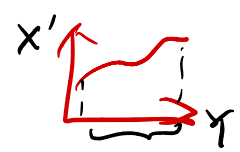
(y', x_2')

Proposition: $A: D(A) \subseteq X \to Y$. If and only if A is densely-defined, $x_1' = x_2'$.

the following set can be viewed as a graph.

$$\Gamma =: \{(y', x') \in Y' \times X' \mid (y', Ax) = (x', x),\ \forall x \in D(A)\}$$

$A': y' \to x'$. $D(A) = P_{Y'}(\Gamma)$.



D(A)

Proof: if $\exists$ (y', x_1'), $(y', x_2') \in Y' \times X'$, $x_1' = x_2'$.

$\forall x \in D(A)$, $(x_1', x) = (y', Ax) = (x_2', x)$. $\Rightarrow$ $(x_1' - x_2', x) = 0$, $\forall x \in D(A)$, $\overline{D(A)} = X$

$\downarrow$

$x_1' = x_2'$.

x

Adjoint of unbounded operator.

$A: D(A) \subseteq H \to H$. A^*: charaterized:

$$\langle Ax, y\rangle = \langle x, A^* y\rangle, \ \forall x \in D(A), \ y \in D(A^*).$$

dual / adjoint:

$$A^* = 6_H' \circ A' \circ 6_H : \quad D(A^*) = 6_H^{-1} \circ D(A') \to H.$$

Riesz Respresentation $\Leftarrow$ $6_H : H \to H' \quad x \mapsto \underbrace{\langle \cdot, x\rangle}$.

$$\Rightarrow \underbrace{(\cdot\ x') = x' = 6 \circ 6^{-1}(x') = \langle \cdot, 6^{-1}(x')\rangle}.$$

$\Leftarrow$: If $D(A)$ is not densely defined, $\overline{D(A)} \neq X$, $\overset{\text{Hahn-Banach}}{\exists}\ x' \neq 0$, $\underbrace{(x', D(A)) = 0}$.

$\downarrow$

$\underbrace{(x', 0) \in P}$.

$\downarrow$

$(0, 0) \in P$.

$\underbrace{(0,0), (x', 0) \in P}$

Closedness, symmetricity, self-adjointness.

$$\begin{tikzcd}
\text{unbounded} \arrow[rr] \arrow[dd] \arrow[dr] & & \text{closedness} \arrow[r, "\text{everywhere}", "\text{defined}"'] \arrow[dd] & \text{bounded} \\
& \text{densely-defined} \arrow[dl] & & \\
\text{symmetric} \arrow[rr] & & \text{self-adjointness} &
\end{tikzcd}$$

Definition. Suppose $A: D(A) \subseteq X \to Y$.

1) A is a closed operator, if the graph $P(A)$ is closed;

2) A is a closable, if A has closed extension;

$\overline{A}$: the smallest closed extension. (closure).

Rmk: A is a closable if and only if $\overline{P(A)}$ is a graph of some operator.

In this case, $\overline{P(A)} = P(\overline{A})$ exactly.

Proof: $\Leftarrow$ if $\overline{P(A)} = P(S)$ for operator S, S must be a closed extension of A;

$\Rightarrow$ if S is a closed extension of A,

$\Rightarrow$ $\overline{P(A)} \subseteq P(S)$, $\Rightarrow$ $\overline{P(A)}$ is a graph of some closed operator R.

$P(S) \supseteq P(R)$.

$\Rightarrow$ S is a abitrary, $\Rightarrow$ $R = \overline{A}$.

Proposition: Suppose $A: D(A) \subseteq X \to Y$.

1) A is closed $\Leftrightarrow$ $\forall\, (x_n, Ax_n) \in P(A)$, $(x_n, Ax_n) \to (x, y)$ implies $x \in D(A)$, $y = Ax$.

2) A is closable $\Leftrightarrow$ $\forall\, (x_n, Ax_n) \in P(A)$, $(x_n, Ax_n) \to (0, y)$ implies $y = 0$.

Proof of 2): $\Rightarrow$ $(0,y)$ lays in $\overline{P(A)} = P(S)$, $y = S0 = 0$ ✓.

$\Leftarrow$ $Sx = \lim Ax_n$, $\forall x \in \overline{D(A)}$. $\{x_n\}$, $\{\bar{x}_n\} \to x$.

(x_n, Ax_n), $(\bar{x}_n, A\bar{x}_n)$ $Sx = \lim Ax_n$,
$= \lim A\bar{x}_n$,

S is a closed extension of A.

$\Rightarrow$ A is closable.

Proposition: $A: D(A) \subseteq H \to H$, H is Hilbert:

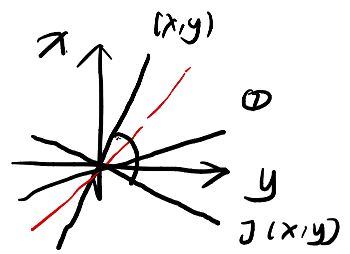
1) if A^* exists, then A^* is closed naturally;

2) if A^{**} exists, then A is closable. In this case, we have:

(A^*, A^{**}) exists, $\overline{D(A)} = \overline{D(A^*)} = H$.

$\bar{A} = A^{**}$, $A^* = (\bar{A})^*$.

Rmk: $\langle Ax, y\rangle = \langle x, \overbrace{A^*y}^{y^*}\rangle$, $\forall x \in D(A)$.



$\Leftrightarrow$ $\langle x, A^*y\rangle + \langle -Ax, y\rangle = 0$, $\forall x \in D(A)$.

$\Leftrightarrow$ $\langle (-Ax, x), (y, y^*)\rangle = 0$, $\forall x \in D(A)$.

$J: (x,y) \to (y,x) \to (-y,x)$.

$(y, y^*) \perp J \circ P(A)$. $J: (x,y) \to (-y, x)$.

$\Leftrightarrow$ $(y, y^*) \in (J\circ P(A))^\perp$. $(x, Ax) \to (-Ax, x)$.

Lemma. A^* exists if and only if $(J\circ P(A))^\perp$ is a graph of some operator $\to A^*$.

In this case, we have $(J\circ P(A))^\perp = P(A^*)$. $(y, y^*) \in P(A^*)$

$\Leftrightarrow$ $(y, y^*) \in (J\circ P(A))^\perp$

Proof of 2): $\overline{\mathcal{P}(A)} = (\mathcal{P}(A)^{\perp})^{\perp} = (J \circ J \circ (\mathcal{P}(A))^{\perp})^{\perp} = (J \circ \mathcal{P}(A^*))^{\perp}$

(if A^{**} exists) $= \mathcal{P}(A^{**})$.

$\downarrow$

$\overline{A} = A^{**}$, $\rightarrow$ $(\overline{A})^* = (A^{**})^* = (A^*)^{**} = \overline{A^*}$.

Theorem. (closed graph). $A : D(A) \subseteq X \to Y$ is closed, bounded iff e.d.

Theorem. (Hille). $A \int = \int A$.

Method to compute the closure of a unbounded operator.

$-\Delta : D(\Omega) \subseteq L^p \to L^p$. X? $-\widetilde{\Delta} : X \subseteq L^p \to L^p$ is closed.

$\downarrow$

Δ? closed C_c^∞

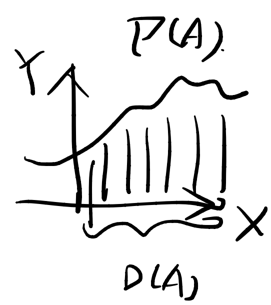
$(u_n, -\Delta u_n)$ $u_n \to u$, L^p

$u_n \in C_c^\infty$ $-\Delta u_n \nearrow$

X, Y complete.

Proposition: $A : D(A) \subseteq X \to Y$ is closed iff $D(A)$ is complete under graph norm;

$\|x\|_P = \|(x, Ax)\|_{X \times Y} = \|x\|_X + \|Ax\|_Y$.



Proof: $\Phi : D(A) \to \mathcal{P}(A)$, $x \to (x, Ax)$.

$\|\cdot\|_P$ $\|\cdot\|_{X \times Y}$.

Φ is isometry. $\|x\|_P = \|(x, Ax)\|_{X \times Y}$.

complete.

$\mathcal{P}(A)$ is closed in $X \times Y$ $\Leftrightarrow$ $\mathcal{P}(A)$ is complete in $X \times Y$ $\Leftrightarrow$ $D(A)$ is complete in $\|\cdot\|_P$.

$\Updownarrow$

A is closed

Application. $A : D(A) \subseteq X \to Y$, $D(\overline{A})$.

A is closable, $\overline{\Phi} : D(\overline{A}) \subseteq X \to \mathcal{P}(\overline{A}) = \overline{\mathcal{P}(A)}$.

$D(\overline{A}) = \overline{D(A)}^{P}$.

$A = -\Delta : C_c^\infty(\Omega) \subseteq L^p \to L^p$ $\mathcal{I}$:

$\|u\|_P = \|u\|_{L^p}^p + \|\Delta u\|_{L^p}^p \sim \|\partial^2 u\|_{L^p}^p$

$\|u\|_{W^{2,p}}$ $\partial u \lesssim \partial^2 u$.

$-\Delta : \overline{C_c^\infty}^{H^2} = H_0^2(\Omega) \subseteq L^p \to L^p$ is closed. closure.

$W_0^{2,p}$

$W^{2,p} \cap W_0^{1,p} \leftrightarrow H^2 \cap H_0^1$.

$A + B \subseteq$

$-\Delta$ self-adjoint extension.

$C_c^\infty(\Omega) \xrightarrow{\text{closure}} H_0^2$

$\downarrow$ self-adjoint

$H^2 \cap H_0^1$

H_0^2: H^2, $u, \partial u|_{\partial\Omega} = 0$

$H^2 \cap H_0^1$: $u|_{\partial\Omega} = 0$.

$H_0^2 \subseteq H^2 \cap H_0^1$

$-\Delta$.

Lame $-\Delta + \overset{\mu}{\nabla}(\nabla\cdot)$

$\downarrow$

$-\Delta$.

closedness

symmetricity $\leftrightarrow$ self-adjointness

(A is symmetric)

Ker A^* is symmetric + A closed $\to$ A is self

$\ker(A^* \pm i) = \{0\}$, $\operatorname{Ran}\{A \pm i\} = X$. $\Rightarrow A$ self

$\sigma(A) \subseteq \mathbb{R}$. $\Rightarrow A$ is self-adjoint. Δ.

$-\Delta$.

$\forall u \in H_0^2 \setminus C_c^\infty$, $\exists u_m \xrightarrow{H^2} u$, .
C_c^∞

$(u_m, -\Delta u_m) \xrightarrow{L^2} (u, -\Delta u)$.

$\downarrow$ $\quad -\Delta u \in L^2$.

$u \in L^2$

$\bar{\in} C_c^\infty$

$\in L^2$

$-\Delta u = f = \lim -\Delta u_m \in L^2$