Review: $A' : y' \mapsto x'$. $\quad A : D(A) \to$ is densely defined.

$$(y', Ax) = (x, \underbrace{A'y'}_{x'}), \quad \forall x \in D(A).$$

$y' \to x'$

$\to x''$

If $\overline{D(A)} \neq X$, $x_0' \perp \overline{D(A)}$, $x_0' \neq 0$.

$$\Rightarrow (x_0', x) = 0, \quad \forall x \in D(A),$$

$$\Updownarrow$$

$$\Leftrightarrow (x_0', x) = (\theta, Ax), \quad \forall x \in D(A).$$

$$\Rightarrow A' : 0 \to x_0' \neq 0$$

$\downarrow$

0

$-\Delta : C_c^\infty \subseteq L^2 \to L^2$.

$-\Delta : H_0^2 \subseteq L^2 \to L^2$. $\quad H_0^2 = \overline{C_c^\infty}^{\|\cdot\|_{H^2}}$

$u|_{\partial\Omega} = 0$.

Neumann

$\frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0$

$-\Delta : D$ $\quad D = \{C^\infty \mid \gamma u = 0\}$

$\overline{D}^{\|\cdot\|_{H^2}} \to \{u \in H^2 \mid \gamma(u, \nabla u) = 0\}$ $\quad N$

$\nabla u \cdot n|_{\partial\Omega}$

$\gamma(u, \nabla u)$

Symmetric

Self-adjoint $\Leftrightarrow$ $\sigma(A) \subseteq \mathbb{R}$.

$-\Delta : H^2 \cap H_0^1 \to$

Lecture 2. invertibility, symmetricity, self-adjointness, spectrum, resolvent

$A : D(A) \subseteq H \to H$.

① A is symmetric if $A \subseteq A^* \Leftrightarrow \forall x, y \in D(A)$, $\langle Ax, y\rangle = \langle x, Ay\rangle$. $\int -\Delta u\, v = \int -v \Delta u$

② A is self-adjoint if $A = A^*$. $D(A^*) \subseteq D(A) \to H$.

Remark: A e.d. symmetric is self-adjoint.

Proposition: If A is (closed) symmetric, $A \subseteq A^{**} \subseteq A^*$. $\bar{A} = A^{**}$

Proof: $A \subseteq \bar{A} \subseteq A^*$, $= A^{**}$. $A \subseteq A^*$, $\bar{A} \subseteq$ closed. $\bar{A} \subseteq A^*$. A^* is closed.

$\Rightarrow A = A^{**} \subseteq A^*$. A^* closed.

$D(A)$, $D(A^*)$ dense $\Rightarrow A^{**}$, A^*. A dense.

$A \subseteq A^*$

$H \supseteq \underset{\text{dense}}{D(A^*)} \supseteq D(A)$

Example: $-\Delta : D(\Omega) \subseteq L^2 \to L^2$. $C_0^\infty \to H_0^1, H^2$

$$\int (-\Delta u) v \overset{u|_{\partial\Omega}=0}{=} \int \nabla u \cdot \nabla v = \int u(-\Delta v). \quad \forall u, v \in D(\Omega) \to H_0^1(\Omega)$$

$-\Delta : D(\Omega)$ self-adjoint extension.

$H^2 \cap H_0^1 = \{H^2 \mid u|_{\partial\Omega} = 0\} \leftarrow H_0^2$

self-adjoint   closure

Self-adjointness:

Proposition. If A is symmetric, then A is self-adjointness if only if only of following conditions holds:

1) $D(A^*) \subseteq D(A)$ ✓ 2) A is closed, A^* is symmetric;

3) A is closed $\ker(A^* \pm i) = 0$, 4) $\mathrm{Ran}(A \pm i) = H$. Δ Cayley transform

5) $\sigma(A) \subseteq \mathbb{R}$.

normal

symmetric

Self-adjoint $\Leftrightarrow$ $\sigma(A) \subseteq \mathbb{R}$.

$\langle Aw, w\rangle = \langle w, Aw\rangle = i\langle Aw, w\rangle$

$\langle Aw, w\rangle$ is real

Spetrum and Resolvent.

Definition: $A: D(A) \subseteq X \to Y$, Banach. H.

$$A^{-1}: R(A) \subseteq Y \to X, \quad A^{-1}y =: x, \text{ if } x = Ay.$$

Rmk: A is injective.

Definition: Invertibility: A is invertible if A^{-1} exists and $\times \to A^{-1}$ everywhere defined

$A \Leftrightarrow \quad \|A^{-1}y\|_X \le C\|y\|, \ \forall y \in R(A) = D^{-1}(A).$

$\Updownarrow$

$L(X,Y)$. $A: X \to Y$.

A^{-1} closed

Proposition: Suppose $A: D(A) \subseteq X \to Y$ densely defined. Then; H

1) $(A')^{-1}$ exists if and only $R(A)$ is dense;

2) if A^{-1} exists, then: $(A^{-1})' = (A')^{-1}$.

$\to (A^{-1})^* = (A^*)^{-1}$

invertible

dual $\Leftrightarrow$ adjoint

Proof: 1) $\Leftarrow$: $A'y'=0 \Rightarrow y'=0.$

$(A'y', x) = (y', Ax) \overset{=0}{,}\ \forall x \in D(A).$

$A'y'=0 \Rightarrow (y', Ax)=0 \Rightarrow (y', R(A))=0 \Rightarrow y'=0.$

$\forall x \in D(A).$

$\Rightarrow$ if $R(A)$ is not dense, $y' \neq 0 \in Y'$. $(y', \overline{R(A)})=0.$

$\Rightarrow (A'y', x) = (y', Ax) = 0.$

$(D(A) = X) \Rightarrow A'y'=0 \Rightarrow A'$

$y' = (A')^{-1}0$

2) $(A^{-1})' = (A')^{-1}$ if A^{-1} exists $\Leftrightarrow$ A is injective.

$(A')^{-1}x \subseteq (A^{-1})'x$. $x' \in D((A')^{-1}) \Rightarrow x' \in D((A^{-1})')$.

$\exists y' \in D(A)$, $A'y' = x'$. $x' \in D(A^{-1})'$:

$\langle\ \rangle$

A^{-1} $\quad (x', A^{-1}y) = (A'y', A^{-1}y) = (y', AA^{-1}y) = (y', y)$, $\forall y \in D(A^{-1})$

$\Rightarrow x' \in D((A^{-1})')$ $\quad (A^{-1})'x' = y' = (A')^{-1}x'$

$(A^{-1})': x' \to y'$

$(A^{-1})' \subseteq (A')^{-1}$. $\quad x' \in D((A^{-1})') \Rightarrow x' \in D((A')^{-1})$

$\langle\ \rangle$

$\exists y' \in Y'$. s.t. $(x', A^{-1}y) = (y', y)$. $\forall y \in D(A^{-1}) = R(A)$

$(A^{-1})'x' = y$

y'

$x \in D(A)$

$\exists x'$. $(y', Ax) = (x', x)$.

$(A^{-1})' \subseteq (A')^{-1}$.

$x' \in (A^{-1})' \Rightarrow x' \in D(A')^{-1} = R(A')$

$\Uparrow$

$\exists y'$, $x' = A'y'$

$\forall x \in D(A)$, $(y', Ax) = (x', A^{-1}(Ax)) = (x', x)$, $x \in D(A)$

$R(A)$, y^A, Ax, y

$A' : y' \mapsto x'$

Definition: Suppose $A : D(A) \subseteq X \to X$, X Banach.

① we define the resolvent of A, denote as $\rho(A)$ ~~dense~~ $(\lambda - A)^{-1}$ exists

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda - A) \text{ is invertible}\}$$

densely defined, $\|(\lambda - A)^{-1} y\| \le C\|y\|$, $y \in D$.

② spectrum $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

$(\lambda - A)^{-1} \in \mathcal{L}(X)$

Rmk: If A is closed, We can show $(\lambda - A)^{-1}$ is c.d. $(\lambda - A)^{-1} \in \mathcal{L}(X)$

Proof of Rmk: $(\lambda - A)^{-1}$ is invertible. $R(A) = R(\lambda - A)$ is dense

$= D((\lambda - A)^{-1})$

$R(\lambda - A)$ is closed. $y_n = (\lambda - A)x_n \to y_0$.

in

$\lambda \in \rho(A) \Rightarrow \|x\| \le \|(\lambda - A)x\|$, $\forall x \in D(A)$

A

$\|x_n\| \le \|(\lambda - A)x_n\| \in H$,

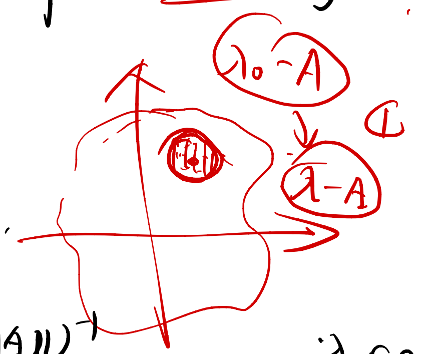
$x_n \to x_0 \in \overline{D(A)}$

$(x_n, (\lambda - A)x_n) \to$

$(x_0, (\lambda - A)x_0)$, $= y_0$

$x_0 \in D(A)$.

Proposition. $A: D(A) \subseteq X \to X$ closed, then $\rho(A)$ is open. λ analytical



Proof: we use the following lemma:

Lemma Δ: Suppose $A: X \to X$ is bounded, with $\|A\|_{\mathcal{L}(X)} < 1$,

then $I - A \in \mathcal{L}(X)$, with $\|(1-A)^{-1}\| \le (1-\|A\|)^{-1}$.

$$I - A = \sum_{n=0}^{\infty} A^n.$$

$\lambda_0 \in \rho(A)$

$(\lambda_0 - A)^{-1} = \sum(\ )$

Proof. Δ.

☆ take $\lambda_0 \in \rho(A)$, $(\lambda - A)^{-1} \in \mathcal{L}(X)$ for any $|\lambda - \lambda_0| < \|(\lambda_0 - A)^{-1}\|^{-1}$.

$$\frac{1}{\lambda - A} = \frac{1}{(\lambda - \lambda_0) + (\lambda_0 - A)} = (\lambda_0 - A)^{-1} \cdot \frac{1}{1 - \frac{\lambda_0 - \lambda}{\lambda_0 - A}} \quad \frac{\lambda_0 - \lambda}{\lambda_0 - A} < 1.$$

$$|\lambda_0 - \lambda|\, \|(\lambda_0 - A)^{-1}\| < 1. \qquad = (\lambda_0 - A)^{-1} \sum_{n=0}^{\infty} \left(\frac{\lambda_0 - \lambda}{\lambda_0 - A}\right)^n$$

$S(\lambda)$

$$S(\lambda) = (\lambda_0 - A)^{-1} \sum (\lambda_0 - \lambda)^n \cdot ((\lambda_0 - A)^{-1})^n \Rightarrow (S(\lambda)) \cdot (\lambda - A) = I.$$

Δ

$$(1 - (\lambda_0 - \lambda)(\lambda_0 - A)^{-1}) \cdot \sum (\lambda_0 - \lambda)^n ((\lambda_0 - A)^{-1})^n = 1.$$

$$S(\lambda) \cdot \underbrace{((\lambda_0 - A) - (\lambda_0 - \lambda))}_{= (\lambda - A)} = I.$$

Proposition:

① (Resolvent formula): $(\lambda-A)^{-1}-(\mu-A)^{-1}=(\mu-\lambda)(\lambda-A)^{-1}(\mu-A)^{-1}$.

② (Resolvent and dual): $((\lambda-A)^{-1})'=(\lambda-A')^{-1}$, $\forall \lambda\in\rho(A)$

$\hookrightarrow$ $\underline{\rho(A)=\rho(A')}$ $\quad (\lambda-A')^{-1}$

$=((\lambda-A)^{-1})'$

Proof: ① $(\lambda-A)^{-1}=(\lambda-A)^{-1}(\mu-A)(\mu-A)^{-1}$

$=(\lambda-A)^{-1}((\mu-\lambda)+(\lambda-A))(\mu-A)^{-1}$

$=(\mu-\lambda)(\lambda-A)^{-1}(\mu-A)^{-1}+(\lambda-A)^{-1}\cdot(\lambda-A)(\mu-A)^{-1}$

② $\lambda\in\rho(A) \Rightarrow$ $(\lambda-A)^{-1}$ is bounded. $\quad \lambda\in\rho(A')$

$\triangle$ $\Downarrow$ $\qquad\qquad \Updownarrow$

$((\lambda-A)^{-1})'=(\lambda-A')^{-1}$ is bound.

is bounded.

✓

$\triangle$. $\lambda\in\rho(A)\Rightarrow \lambda\in\rho(A')$. ✓

$\Leftarrow$ ✗

☆ $\Leftarrow$ $A: D(A)\subseteq X\to Y$, if $R(A')$ is weakly $*$ dense,

$\Downarrow$ A has inverse

$\lambda\in\rho(A')$. $\quad (\lambda-A')^{-1}\in L(X) \Rightarrow R(\lambda-A')$ is dense.

$\Downarrow$

$\lambda\in\rho(A)$ $\quad \Rightarrow \lambda-A$ has a inverse.

Lemm.