Lecture 3.

① $A \rightarrow \quad \rho(A) = \rho(A')$. A is closed.

✓ $\rho(A) \subseteq \rho(A')$. $\longrightarrow$ $\lambda \in \rho(A) = \rho(A')$ $\quad ((\lambda - A)^{-1})' = (\lambda - A')^{-1}$. ✓

$\rho(A') \subseteq \rho(A)$.

② (symmetric). self-adjoint $\Leftrightarrow \sigma(A) \subseteq \mathbb{R}$.

normal.

③ Friedrich's extension.

④ two characterization.

Proposition. Suppose $A: D(A) \subseteq X \to Y$ is densely defined. Then.

✓ 1) $(A')^{-1}$ exists if and only if $R(A)$ is dense;

✓ 2) moreover, A^{-1} exists, then $(A^{-1})' = (A')^{-1}$.

3) A^{-1} is bounded if and only if A is closed and $(A')^{-1}$ is bounded.

σ^{-1} compact

Proof of 3). A^{-1} is bounded $\Rightarrow$.

A' is bounded implies $(A^{-1})' = (A')^{-1}$ is bounded.
closed.

$\Leftarrow$. $(A')^{-1}$ is bounded $\Rightarrow$ A^{-1} is bounded.

$x \in R(A)$.

$x \in X$.

$$|(A^{-1}x, x')| = |(x, (A^{-1})'x')| = |(x, (A')^{-1}x')| \le \|(A')^{-1}\| \, \|x'\| \, \|x\|$$

Since $R(A)$ is dense, A is closed $\Rightarrow$ A^{-1} is bounded.

Proof: $\rho(A') \subseteq \rho(A)$. $\lambda \in \rho(A') \Rightarrow \lambda \in \rho(A)$.

$\lambda \in \rho(A') \Rightarrow (\lambda - A')^{-1}$ { densely defined, bounded } $\Rightarrow R(\lambda - A') = D((\lambda - A')^{-1})$ is dense

$\Downarrow$ $\qquad\qquad\qquad\qquad \Downarrow$

$\overline{R(\lambda - A)} = X \qquad\qquad (\lambda - A)^{-1}$ exists

A

$(\lambda - A)x_0 = 0. \qquad x_0 \neq 0$

$(x_0, (\lambda - A)' y) = ((\lambda - A) x_0, y) = 0$, $(\lambda-A)x_0 = 0$

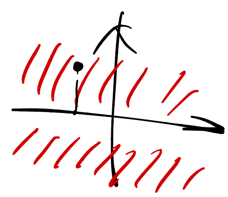
$\neq 0$ $\qquad y \in D(A')$

$R((\lambda - A)')$ is not dense.

Proposition 1. Suppose $A : D(A) \subseteq X \to X$ is symmetric, then A is self-adjoint $\Leftrightarrow \sigma(A) \subseteq \mathbb{R}$

Proof: ① self-adjoint $\Rightarrow$ symmetric $+ \ \sigma(A) \subseteq \mathbb{R}$.

$\Leftrightarrow \mathbb{R}^c \subseteq \rho(A)$.



$\mathbb{R}^c$ $\quad |\mathrm{Im}\,\lambda| \neq 0$

$\Leftarrow \lambda - A$ is bijective. $\qquad$ invertible

$(\lambda - A)^{-1} \in L(X)$ $\qquad$ closed + everywhere

injective: $v = (\lambda - A)u$

$\langle v, u \rangle = \lambda \overset{>0}{\langle u, u \rangle} - \langle Au, u \rangle$ $\quad$ is real

$\langle Au, u \rangle = \langle u, Au \rangle = \overline{\langle Au, u \rangle}$

$\mathrm{Im} \langle v, u \rangle = \mathrm{Im}\,\lambda \overset{\neq 0}{\langle u, u \rangle}$

$\|u\| \|v\| \geq |\mathrm{Im}\,\lambda| \, \|u\|^2 \Rightarrow \|u\| \leq \dfrac{\|v\|}{|\mathrm{Im}\,\lambda|}$

$\|(\lambda - A) u\|$ $\qquad (\lambda - A)u$

$(\lambda - A)$ Subjective: closed. $Au = \lambda u + v$. $(\lambda - A)u_n$ is closed

$R(\lambda - A)$ is closed

$\Downarrow$ u_n (Au_n, u_n)

densely $\downarrow$ A closed

$(\lambda - A)u \to$ (Au, u)

$\overline{R(\lambda - A)} = X$: $\exists\, w \neq 0$,

$\langle (\lambda - A)u, w \rangle = 0$.

$\Rightarrow \langle u, \bar\lambda w \rangle = \langle \lambda u, w \rangle = \langle Au, w \rangle = \langle u, Aw \rangle$

$\Rightarrow \langle Aw, w \rangle = \bar\lambda \langle w, w \rangle$ is not real

real $\nrightarrow$.

② (symmetric). $\sigma(A) \subseteq \mathbb{R} \Rightarrow$ self-adjoint.

$\mathbb{R}^c \subseteq \rho(A) \Rightarrow \lambda \in \rho(A) \Leftrightarrow \bar\lambda \in \rho(A)$ $\qquad \lambda \in \rho(A) \Rightarrow \bar\lambda \in \rho(A)$

λ $\qquad (\bar\lambda - A)^{-1}$

$\langle \underbrace{(\lambda - A)y}_{x}, z \rangle = \langle y, \underbrace{(\bar\lambda - A)z}_{s} \rangle$, $\forall y, z$ $\qquad R(\bar\lambda - A) = X$, $R(\lambda - A) = X$

$\Leftrightarrow \langle x, (\bar\lambda - A)^{-1} s \rangle = \langle (\lambda - A)^{-1} x, s \rangle$, $\forall x \in R(\lambda - A) = X$

$\Rightarrow s = x$

$D(A^*) \subseteq D(A)$

$v \in D(A^*)$,

$\lambda \in \rho(A)$

$\langle v, x \rangle = \langle v, (\lambda - A)(\lambda - A)^{-1} x \rangle$

$= \langle (\bar\lambda - A^*)v, (\lambda - A)^{-1} x \rangle$

$= \langle (\bar\lambda - A)^{-1}(\bar\lambda - A^*)v, x \rangle$ $\qquad x \in X$

$\Rightarrow \; v = (\bar\lambda - A)^{-1}(\bar\lambda - A^*)v \in R((\bar\lambda - A)^{-1})$

$= D(\bar\lambda - A)$

$D(A^*) \subseteq$ $= D(A)$.

$H_0^2 \cap H_0^1$

$-\Delta : D \subseteq L^2 \to L^2$. H_0^2

$A \subseteq \tilde A$, $\tilde A$ is self-adjoint

Friedrich self-adjoint extension.

Proposition: A semibounded symmetric operator $A : D(A) \subseteq H \to H$ has a self-adjoint extension $\tilde A$ with same lower bound.

lower bound.

Definition. $A : D(A) \subseteq H \to H$ is semibounded if $\exists c \in \mathbb{R}$. $\langle Au, u\rangle \geq c\langle u, u\rangle$, $\forall u \in D(A)$

Proof: We consider $c = 1$ case.

$C = 0$

$\langle Au, u\rangle \geq 0$.

$A \Rightarrow A + (1-c)I \Rightarrow \widetilde{A + (1-c)I} \Rightarrow A + (1-c)I + (c-1)I \supseteq A$

c $\qquad 1$ $\qquad D(\tilde A)$

$\langle Au, u\rangle \geq \langle u, u\rangle$

Consider: $\langle u, v\rangle_{D(A)} =: \langle u, Av\rangle_H$, $\forall u, v \in D(A)$. $D(A)$

$\langle u, u\rangle_{D(A)} = \langle Au, u\rangle \geq \langle u, u\rangle_H > 0$ $\quad u \neq 0$

$\langle u, v\rangle_{D(A)} = \overline{\langle v, u\rangle}_{D(A)}$ $\quad$ semibounded

symmetric

$\langle u, v\rangle_{D(A)} = \langle u, Av\rangle_H = \langle Au, v\rangle = \overline{\langle v, Au\rangle}$

$= \overline{\langle v, u\rangle}_{D(A)}$

$V = \overline{D(A)}^{D(A)}$

$H \supseteq V \supseteq D(A)$ is dense $\quad H$

V is dense in H.

$\Rightarrow \quad D(\tilde{A}) = \{ u \in V \mid \exists v \in H, \text{ s.t. } \langle \cdot, u \rangle_V = \langle \cdot |_V, v \rangle_H \}$ $\quad v_1 = v_2$

$\Leftrightarrow \quad \langle \cdot, u \rangle_V$ is bounded functional in $(V, \langle \cdot, \cdot \rangle_H)$

$\tilde{A} : u \mapsto v \qquad u \in D(\tilde{A})$

$\exists C$

$\Leftrightarrow \quad \langle w, Au \rangle \le C \|w\|, \forall w \in H$

$v_1, v_2,$

$\langle w, v_1 \rangle_H = \langle w, v_2 \rangle_H \quad \in V \quad w \in V$

$\forall w \in H \qquad V, H \qquad Lu$

$\Rightarrow v_1 = v_2$

$\langle (-\Delta) u, v \rangle$

$\langle Du, Dv \rangle$

$-\Delta u \to Lu$

$\tilde{A} \begin{cases} \text{extension: } \tilde{A} \supseteq A \quad ① \\ \text{self-adjoint} \quad ② \end{cases}$

$\langle iAu, w \rangle$

$= \langle i$

① $\tilde{A} \supseteq A$. $u \in D(A)$,

$\langle \cdot, u \rangle_V = \langle \cdot |_V, Au \rangle_H \Rightarrow \tilde{A} u = Au$

$v = \tilde{A} u$ $\qquad$ $\tilde{A}$ is surjective

$\langle w, u \rangle_V = \langle w, Au \rangle_H, \forall w \in V, u \in D(A)$

② $\tilde{A}$ is symmetric $\Rightarrow (\tilde{A})^{-1}$ is symmetric $\qquad (\tilde{A})^{-1}$ is everywhere defined.

$\Rightarrow (\tilde{A})^{-1}$ is self-adjoint $\qquad D(A) \subseteq D(A^*)$

$\Rightarrow \tilde{A}$ is self-adjoint. $\qquad D(A^*) \subseteq D(A) = X$

$X = Y$

bijective

1) $\tilde{A}$ is surjective: $\forall v \in H$, $\exists u \in V$, $\langle \cdot, v\rangle_H = \langle \cdot, u\rangle_V$.

$|\langle w, v\rangle_H| \le \|w\|_H \|v\|_H \le \|w\|_V \|v\|_H$. H. $(V, \|\cdot\|_H)$

$V \hookrightarrow H$. $\tilde{A}$ $\langle u, Au\rangle \ge \langle u, u\rangle$ A 1-semibounded

$\exists u \in V$, $\langle \cdot, u\rangle_V = \langle \cdot, v\rangle_H$. Riesz: $(V, \|\cdot\|_V)$

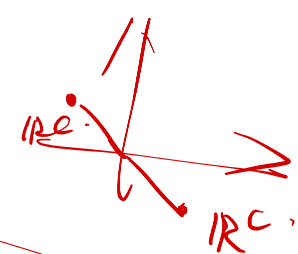
$\tilde{A}$ is symmetric. I. V. V.

$$\langle w, \tilde{A}u\rangle_H = \langle w, u\rangle_V = \overline{\langle u, w\rangle_V} = \overline{\langle u, \tilde{A}w\rangle_H}$$
$$= \langle \tilde{A}w, u\rangle_H, \ \forall u, w \in D(\tilde{A})$$

$(\tilde{A})^{-1}$ symmetric.

$$\langle \tilde{A}^{-1}u, v\rangle = \langle \tilde{A}^{-1}u, \tilde{A}\tilde{A}^{-1}v\rangle = \langle \tilde{A}\tilde{A}^{-1}u, \tilde{A}^{-1}v\rangle$$
$$= \langle u, \tilde{A}^{-1}v\rangle.$$

A^{-1} is self-adjoint $\Rightarrow$ A is self-adjoint.



$\sigma(A^{-1}) \subseteq \mathbb{R} \Rightarrow \sigma(A) \subseteq \mathbb{R}$

$\lambda \in \mathbb{R}^c$, $(\lambda - \tilde{A}^{-1})^{-1}$ is bounded

$\Rightarrow (\lambda^{-1} - \tilde{A})^{-1} = -\lambda(\lambda - \tilde{A}^{-1})^{-1}\tilde{A}^{-1}$ is bounded

$\Rightarrow \lambda^{-1} \in \mathbb{R}^c \Rightarrow \sigma(A) \subseteq \mathbb{R}$.

$\Rightarrow$ $\lambda \in \mathbb{R}^c$ $(\lambda - \hat{A})^{-1}$ is bounded

Δ symmetric

$i\Delta$ is not sym

$$\int (-i\Delta u)\bar{v} = \int i\,\nabla u \cdot \overline{\nabla v} = \int (u \cdot \overline{(i\Delta v)}).$$

i $-i$ $\bar{i}$

Ω

$-\Delta : D(\Omega) \subseteq L^2 \to L^2$

$-\Delta \geq 0$ $\quad \int -\Delta u\, u = \int (\nabla u\, \nabla u) \geq 0$

$D(-\Delta)$

$V = \overline{D(\Omega)}^{H^1} = H^1_0 \cap H^2$

$\geq \int |u|_{L^2}$

$D(-\Delta + I) = \{u \in H^1_0 \mid (\Delta + I) \text{ is bounded}$ in $(H^1_0, \|\cdot\|_{L^2})$ $-\Delta + I$

$\|$

$H^1_0 \cap H^2$

$u \in H^2$ $-\Delta u \in L^2$ $\checkmark$

$\langle u, v\rangle_{D(A)} = \langle (-\Delta + 2)u, v\rangle_{L^2}$

$= \int u u + \int \nabla u \cdot \nabla u$

$\Rightarrow \|u\|_{D(A)} = \|u\|_{H^1}$