Lecture 0. A brief introduction of unbounded operators.

In this lecture, I'd like to introduce the basic notions about (unbounded) operators $A: D(A) \subseteq X \to Y$. Indeed, the most main result why we investigate the unbounded case for PDEers is that the differential operators are always unbounded.

> Example 1. We consider the differentiation
> $$A = \frac{d}{dt} : C^1 \subseteq C[a,b] \longrightarrow C[a,b].$$
> Then let $u_n(t) = \sin nt$, then $\|u_n\|_{C[a,b]} = 1$, but $\|Au_n\|_{C[a,b]} = n \to \infty$, which implies $A = \frac{d}{dt}$ is unbounded.

So we'd like to formulate a rigorous theory for the unbounded operators, which will serve as a fundamental language for PDE. It can be introduced briefly as following parts:

1) Basic notions: the extension and dual/adjoint of a unbounded operators. The closed, symmetric and self-adjoint operators and its related facts. Some applications to the Laplacian $-\Delta$, including its closure and self-adjoint extensions.

2) Spectral theory: The resolvent and spectrum for unbounded operators. Two classification of spectrum set. spetrum of $-\Delta$.

Functional calulus: $A = \int_{\sigma(A)} \lambda \, dE(\lambda) \Rightarrow f(A) = \int_{\sigma(A)} f(\lambda) \, dE(\lambda)$.

Roughly saying, we can characterization the operators by its spectrum. And to obtain a kind of operator, we can operate directly on the spetrum place and return the operation back to the operator level.

3) Perturbation: We investigate the stability of the functional properties, including the closedness, invertibility, self-adjointness and the spectrum set

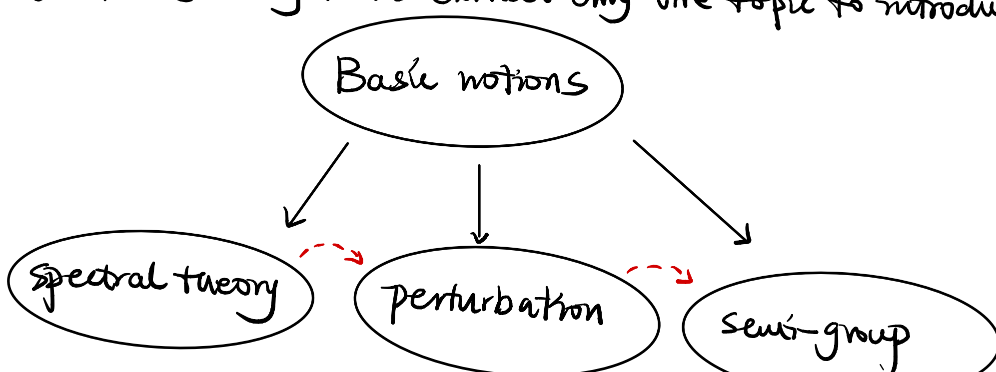
under a proper perturbation. Indeed, many kind of operators we concerned, such as Schrodinger operator $-\Delta + V$, stokes operator $-P\Delta$ and Lame operator $-\Delta + \nabla(\nabla\cdot)$, can be viewed as perturbations of Laplacian $-\Delta$.

4) Semigroup theory: we rigorously define the semigroup e^{tA}, and analyse when we can generate such kind of a semi-group.
$\begin{cases} \text{Hille–Yosida,} \\ \text{Lumer–Philips;} \\ \text{Stone} \end{cases}$

Using such concepts, we can strictly establish the well-posedness for abstract Cauchy problems: $\begin{cases} \partial_t u - Au = f \\ u(0) = u_0. \end{cases}$ (The Navier-Stokes and Schrodinger can be reduced to such kind of abstract case.)

As for more deep topic I'd mentioned, it's better to introduce them after we finish the above fundament theorys. (pesudo-spetrum, analytic semigroups, spectral anlysis)

I should mention that the basic notions in spectral \ perturbation \ semigroups are independent, are many books extract only one topic to introduce:



except some advanced results. I'd like to clarify this point for a clear sketch in mind.

Lecture 1. Basic notions about unbounded operator.

As we introduce the differential operator $A=\frac{d}{dt}$, the domain of unbounded operators are not always everywhere-defined:

$$A: D(A) \subseteq X \to Y.$$

For example, the Laplacian $-\Delta: D \subseteq L^p \to L^p$ defined naturally in D. But we may concern its extension in weak sense. (for example, in sobolev spaces).

Definition 1. We define the graph of a operator A as:

$$\Gamma(A) = \{(x, Tx) \in X \times Y \mid x \in D(A)\}$$

Then we say $\tilde{A}$ is a extension of A, denoted as $A \subseteq \hat{A}$, if $\Gamma(A) \subseteq \Gamma(\hat{A})$.

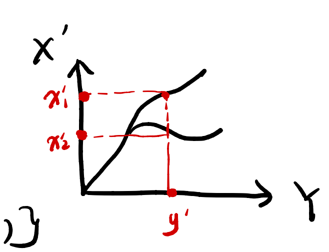
Remark. It is equivalent to say, for $x \in D(A)$, we have $x \in D(\hat{A})$ and $Ax = \tilde{A}x$.

The another important concept is the dual (or conjugate, in some text book) of a unbounded operator (The analogue one is the adjoint for Hilbert case). We recall that for bonded operator $A: X \to Y$, the dual operator $A': Y' \to X'$ is defined by following charcterization:

$$(Ax, y') = (x, A'y'), \ \forall x \in X, \ y' \in Y'.$$

The similar idea is applied to the unbonded operators, but a well-definedness argument should be noted due to not-everywhere-defined domain. And this is the reason why we always require dense-definedness when the dual arises.

Proposition 1. Let X, Y be N.V.S., and X', Y' be the dual. Consider $A: D(A) \subseteq X \to Y$. Then if and only if A is densely defined, the following set can be viewed as a graph



$$\Gamma =: \{(y', x') \in Y' \times X' \mid (y', Ax) = (x', x), \forall x \in D(A)\}$$

for some operators. In this case, we define $A': y' \mapsto x'$, $D(A') = P_{Y'}(\Gamma)$.

Remark. In the dense-defined case, A' exists and is characterized by

$$(y', Ax) = (A'y', x), \quad \forall x \in D(A),\ y' \in D(A'). \quad ☆$$

Proof. That is to show, given if there exists (y', x_1'), (y', x_2') lays in P, then $x_1' = x_2'$. Indeed,

$$(x_1', x) = (y', Ax) = (x_2', x), \ \forall x \in D(A), \Rightarrow (x_1' - x_2', D(A)) = 0.$$

Since $D(A)$ is dense in X, then $x_1' = x_2'$.

Adjoint of a operator

Suppose $A: D(A) \subseteq H \to H$, where H is a Hilbert space. We can similarly define the adjoint of A if A is densely defined. And it is charaterized by:

$$\langle Ax, y \rangle = \langle x, A^* y \rangle, \ \forall x \in D(A),\ y \in D(A^*).$$

We show notice that the adjoint and the dual has the following relation:

$$A^* = 6_H^{-1} \circ A' \circ 6_H : D(A^*) = 6_H^{-1} \circ D(A') \to H$$

Proposition. Suppose $A: D(A) \subseteq H \to H$, where H is a Hilbert space, then:

1) the dual A^* exists if and only if A is densely defined;

2) under the above case, if further $A^{**} = (A^*)^*$ exists, then $A \subseteq A^{**}$ naturally

Proof: 1) The idea is analogue to the dual case and so we omit it.

2) For $x \in D(A)$, we show that $x \in D(A^{**})$ as there exists unique (grant by existence) $x^* = Ax$ such that $\langle A^* y, x \rangle = \langle y, Ax \rangle = \langle y, x^* \rangle$, $\forall y \in D(A^*)$.

Consequently, we see $A^{**}x = (A^*)^* x = x^* = Ax$, which implies $A \subseteq A^{**}$.

Closedness, symmetricity and the self-adjointness

In this part, we'd like to introduce the notions about closedness and symmetricity, and finally establish the basic facts about self-adjoint operators. The mind trace can be formulated as following digram:

$$
\begin{tikzcd}
\text{unbounded} \arrow[rr] \arrow[dd] \arrow[dr] & & \text{closed} \arrow[r, "\text{everywhere}", "\text{defined}"'] \arrow[dd] & \text{bounded} \\
& \text{densely defined} \arrow[dl] & & \\
\text{symmetric} \arrow[rr] & & \text{self-adjointness} &
\end{tikzcd}
$$

First we'd like to define the relevant concepts about closedness:

> **Definition 2.** Suppose $A: D(A) \subseteq X \longrightarrow Y$. Then we say A is a closed operator if the graph $\Gamma(A)$ is closed. More generally, A is closable if it has a closed extension. In this case, we define the smallest closed extension as the closure of A, denoted as $\bar{A}$.

Remark: There is a more useful definition to characterize the closability:

> A is closable if and only if $\overline{\Gamma(A)}$ is a graph of some operator, in this case $\overline{\Gamma(A)} = \Gamma(\bar{A})$ exactly.

Indeed, we can show the equivalence:

$\Leftarrow$: if $\overline{\Gamma(A)} = \Gamma(S)$ for some operator S, then S must be a closed extension of A;

$\Rightarrow$: if S is a closed extension of A. Then by definition $\overline{\Gamma(A)} \subseteq \Gamma(S)$, which implies $\overline{\Gamma(A)}$ is uniquely correspondent ( ) and thus a graph of some closed operator, denoted as R. The abitrary of S implies $R = \bar{A}$.

Some clarification are listed below:

Proposition. Suppose $A: D(A) \subseteq X \to Y$. Then

1) A is closed if and only if $\forall (x_n, Ax_n) \in P(A)$, $(x_n, Ax_n) \to (x, y)$ implies $x \in D(A)$ and $y = Ax$.

2) A is closable if and only if $\forall (x_n, Ax_n) \in P(A)$, $(x_n, Ax_n) \to (0, y)$ implies $y = 0$.

Proof: 1) Direct from the definition.

2) If A is closable, then the limit $(0, y)$ lays in $\overline{P(A)} = P(S)$ for some linear operator S, which implies $y = S(0) = 0$.

Conversely, we can define the operator S by following approximation:

$$Sx = \lim Ax_n, \quad \forall x \in \overline{D(A)},$$

there is well-defined by the assumption (free with choice of Ax_n).

Now we discuss the properties when it involves the adjoint.

Proposition: suppose $A: D(A) \subseteq H \to H$, where H is Hilbert:

1) iff A^* exists, then A^* is closed naturally;

2) iff further A^{**} exists, then A is closable. In this case, we have:

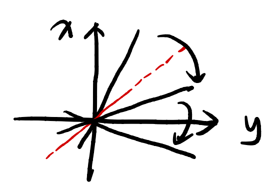
$$\bar{A} = A^{**}, \quad A^* = (\bar{A})^*.$$

Indeed, we can observe from the characterization:

$$\langle Ax, y \rangle = \langle x, A^* \overset{y^*}{y} \rangle, \quad \forall x \in D(A),$$

$$\Leftrightarrow \quad \langle (-Ax, x), (y, y^*) \rangle = 0, \quad \forall x \in D(A).$$

$$\Leftrightarrow \quad (y, y^*) \in (J \circ P(A))^\perp, \quad J: (x, y) \mapsto (-y, x).$$



Then we can easily obtain the following lemma:

Lemma: T^* exists if and only if $(J \circ P(A))^\perp$ is a graph of some operator. In this case, we have $(J \circ P(A))^\perp = P(T^*)$.

Proof of proposition: 1) It is directly from the lemma, since compliment is always closed.

2) we should notice that: $\overline{P(A)} = (P(A)^{\perp})^{\perp} = (J \circ J \circ P(A)^{\perp})^{\perp} = (J \circ P(A^*))^{\perp}$ (if A^{**} exists) $= P(A^{**})$.

invariant for linear space

This implies, A is closable, and its closure is exactly A^{**}. ($\bar{A} = A^{**}$)

and $(\bar{A})^* = (A^{**})^* = (A^*)^{**} = \overline{A^*}$.

Converse side check later. Δ.

Some other properites will be checked later when we need.

Theorem. (Closed Graph). Suppose $A: D(A) \subseteq X \to Y$ is closed, then A is bounded iff e.d.

Theorem. (Hille). $A \int = \int A$.

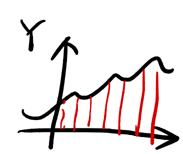
Method to compute a closure.

Here we'd like to give a method to compute the closure extension, and take the Laplacian $-\Delta$: $D \subseteq L^p \to L^p$ for example.

> **Proposition:** Suppose X, Y is Banach, then $A: D(A) \subseteq X \to Y$ is closed if and only if $D(A)$ is complete under the graph norm $\|\cdot\|_P$ defined by
> $$\|x\|_P = \|(x, Ax)\|_{X \times Y} = \|x\|_X + \|Ax\|_Y.$$

Proof: We consider the isometry:

$$\Phi: D(A) \to P(A), \quad x \mapsto (x, Px)$$
$$\begin{array}{cc} \downarrow & \downarrow \\ \|\cdot\|_P & \|\cdot\|_{X \times Y} \end{array}$$



$P(A)$ is closed in $X \times Y \Leftrightarrow P(A)$ is complete in $X \times Y \Leftrightarrow D(A)$ is complete in $\|\cdot\|_P$

Then the closure of $-\Delta$: $D \subseteq L^2 \to L^2$ has the domain:

$$D(\overline{-\Delta}) = \overline{D(-\Delta)}^{P} = \overline{D(-\Delta)}^{H^2} = \overline{D}^{H^2} = H_0^2.$$

Since $\|u\|_P = \|u\|_{L^2} + \|\Delta u\|_{L^2} \overset{u \in D}{\sim} \|u\|_{H^2}$ by Poincare inequality.

Symmetricity and self-adjointness.

As the adjoint notions are always involved in this section, we'd like to assume $A: D(A) \subseteq H \to H$ is densely defined first.

Definition: We say $A: D(A) \subseteq H \to H$ is symmetric if $A \subseteq A^*$, self-adjoint if $A = A^*$. And say A is essentially self-adjoint if $\bar{A}$ is self-adjoint.

We'd like to clarify these notions in later discussion. Indeed, a symmetric operator A can be charaterized by:

$$\langle Ax, y \rangle = \langle x, Ay \rangle, \quad \forall x \in D(A),\ y \in D(A).$$

Then adjoint relation of a symmetric is quite interesting:

Proposition: Suppose A is a (closed) symmetric operator. Then A^*, A^{**} exists and $A \subseteq A^{**} \overset{(=)}{\subseteq} A^*$.

Proof: A^{**} exists since $A \subseteq A^*$ implies $D(A^*) \supseteq D(A)$ is dense. In this case, we know that A is closable and $A \subseteq \bar{A} = A^{**}$. Moreover, since A^* is closed, then $\bar{A} \subseteq A^*$, which grant finally $A \subseteq \underset{A^{**}}{\underset{\|}{\bar{A}}} \subseteq A^*$.

The following proposition give several equivalent condition when a symmetric operator is self-adjoint.

Proposition. Suppose $A: D(A) \subseteq H \to H$ is a symmetric operator, then the following statements are equivalent:

1) A is self-adjoint; 2) $D(A) = D(A^*)$; 3) A is closed and A^* is symmetric;

4) A is closed and $\operatorname{Ker}(* \pm i) = \{0\}$; 5) $\operatorname{Ran}(A \pm i) = X$.

6) $\sigma(A) \subseteq \mathbb{R}$. (prove later).

Proof: 1) $\Leftrightarrow$ 2) is clearly from the definition.

1) $\Leftrightarrow$ 3) is from the fact: $A^* \subseteq (A^*)^* = \bar{A} = A$.

1) $\Rightarrow$ 4) Self-adjoint implies $A = A^*$ is closed clearly, moreover, $\forall x \in \ker(A^* \pm i)$

$(A^* \pm i)x = 0$,

$\mp i \langle x, x\rangle = \langle \mp i x, x\rangle = \langle Ax, x\rangle = \langle x, Ax\rangle = \langle x, \mp i x\rangle = \pm i \langle x, x\rangle$

$\Rightarrow \langle x, x\rangle = 0 \Rightarrow x = 0$.

4) $\Rightarrow$ 5). clearly $\overline{\operatorname{Ran}(T \pm i)} = X$, we show than $\operatorname{Ran}(T \pm i)$ is closed.

The important observation is that

$$\|(T \pm i)x\|^2 = \langle (T \pm i)x, (T \pm i)x\rangle$$
$$= \langle Tx, Tx\rangle + \langle x, x\rangle + \underbrace{\langle Tx, \pm i x\rangle + \langle \pm i x, Tx\rangle}_{=0}$$
$$= \|Tx\|^2 + \|x\|^2.$$

Then for $\{(T \pm i)x_n\}$ cauchy, so is $\{x_n\}$ and $\{Tx_n\}$. and closedness ,

ensure $x_n \to x_0$ and $Tx_n \to y_0$, which implies: $(T \pm i)x_n \to (T \pm i)x_0$.

5) $\Rightarrow$ 1). We need to show for $x \in D(A^*)$, $x \in D(A)$. Since $\operatorname{Ran}(A \pm i) = X$,

we can find $x' \in D(A)$ such that $(A \pm i)x' = (A^* \pm i)x$. Following we show $x = x'$.

$\triangle$ $A \subseteq A^* \Rightarrow (A \pm i)x' = (A^* \pm i)x' \Rightarrow (A^* \pm i)(x' - x) = 0 \Rightarrow x' - x = 0$.

A direct corollary from 4), 5) is the following characterization for essentially self-adjoint:

Corollary: A symmetric operator $A: D(A) \subseteq H \to H$ is essentially self-adjoint iff

1) $\ker(A^* \pm i I) = 0$; or 2) $\overline{\operatorname{Ran}(A \pm i I)} = H$.