# Schauder Theory.

## Introduction:

Dirichlet Problem:

$$\begin{cases} -a^{ij}D_{ij}u + b^i D_i u + cu = f, & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases}$$

① $\lambda |\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall x, \xi \in \mathbb{R}^n \quad \Lambda \ge \lambda > 0$

② $a^{ij}, b^i, c \in C^{0,\alpha}(\bar{\Omega})$

$$\longrightarrow \begin{cases} -a_{ij}(x_0)D_{ij}u = \tilde{f} = f - b^i D_i u - cu - (a_{ij}(x_0) - a_{ij}(x))D_{ij}u \\ u = \varphi. \end{cases}$$

$$\frac{1}{\lambda}\left\{ \sum_{ij}|a^{ij}|_{\alpha;\Omega} + \sum_i |b^i|_{\alpha;\Omega} + |c|_{\alpha;\Omega} \right\} \le \Lambda$$

$\partial\Omega$ smooth. $c \ge 0$ $\to$ Maximum Principle. $f \in C^{0,\alpha}(\bar{\Omega})$. $\varphi \in C^{2,\alpha}(\bar{\Omega}$

$-\Delta u + cu = 0$.

$(D_{ij}u)$ negative definite $\quad \Delta u < 0 \quad -\Delta u > 0$.

1. Hölder Space.

2. Molifier Kernel

Hölder norm

$u \in C^{0,\alpha} \quad \tilde{u}(x,\tau)$

$$\boxed{[u]_\alpha \sim \sup_{\tau > 0, x \in \mathbb{R}^n} \tau^{1-\alpha}|D\tilde{u}(x,\tau)|}$$

3 Potential equation

$-\Delta u = f$.

$[D^2u]_\alpha \le C[f]_\alpha$.

$$[u]_\alpha = \frac{|u(x)-u(y)|}{|x-y|^\alpha} = |Du(x+\theta(y-x))|\,|x-y|^{1-\alpha}$$

Schauder interior estimate.

4. $Lu = -a^{ij}D_{ij}u + b^i D_i u + cu = f \quad$ in $\Omega$

$$|u|_{2,\alpha;\Omega'} \leq C\left(\frac{1}{\lambda}|f|_{\alpha;\Omega} + |u|_{0;\Omega}\right).$$

5. Schauder Global estimation.

$$\begin{cases} Lu = -a^{ij}D_{ij}u + b^i D_i u + cu = f & \text{in } \Omega \\ u = \varphi \end{cases}$$

$\varphi \in C^{2,\alpha}(\bar{\Omega})$. $f \in C^{0,\alpha}(\bar{\Omega})$.

$$\boxed{|u|_{2,\alpha;\Omega} \leq C\{|f|_{\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega} + |u|_{0;\Omega}\}.}$$

## §1 Hölder Space

$H_{x_0}^{\alpha}[u;\Omega] = \sup\limits_{\substack{x\in\Omega \\ x\neq x_0}} \frac{|u(x)-u(x_0)|}{|x-x_0|^{\alpha}} < \infty$ Then we say $u$ is Hölder countinuous with index $\alpha$ at $x_0$.

Def 1. $k \in N^*$, $0<\alpha\leq 1$.

$$[u]_{\alpha;\Omega} = \sup_{x_0\in\Omega} H_{x_0}^{\alpha}[u;\Omega] = \sup_{\substack{x\neq y \\ x,y\in\Omega}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}.$$

$$[u]_{k;\Omega} = \sum_{|\alpha|=k} \sup_{x\in\Omega} |D^{\alpha}u(x)|.$$

$$[u]_{k,\alpha;\Omega} = \sum_{|\beta|=k} \sup_{x\in\Omega} [D^{\beta}u]_{\alpha;\Omega}$$

$C^k(\bar{\Omega})$: $u\in C^k(\bar{\Omega})$. $\|u\|_{k;\Omega} = \sum\limits_{i=1}^{k} [u]_{k;\Omega} < \infty$.

$C^{k,\alpha}(\bar{\Omega})$: $u\in C^k(\bar{\Omega})$. $\|u\|_{k,\alpha;\Omega} = \|u\|_{k;\Omega} + [u]_{k,\alpha;\Omega} < \infty$

$[C^k(\bar{\Omega}), \|\cdot\|_{k;\Omega}]$, $[C^{k,\alpha}(\bar{\Omega}), \|\cdot\|_{k,\alpha;\Omega}]$ Banach Space.

Pro 1. $u, v \in C^{0,\alpha}(\bar{\Omega})$, $\Rightarrow$ $[uv]_{\alpha} \leq [u]_{\alpha}[v]_0 + [u]_0[v]_{\alpha} \leq \|u\|_{0,\alpha}\|u\|_{0,\alpha}$.

Proof: $[uv]_{\alpha} = \sup\limits_{x,y\in\Omega} \frac{|u(x)v(x)-u(y)v(y)|}{|x-y|^{\alpha}}$

$$\because \frac{|u(x)v(x)-v(y)u(y)|}{|x-y|^\alpha} = \frac{|u(x)(v(x)-v(y)) + v(y)(u(x)-u(y))|}{|x-y|^\alpha}$$

$$\le |u(x)| \frac{|v(x)-v(y)|}{|x-y|^\alpha} + |v(y)| \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

$$\Rightarrow [uv]_\alpha \le [u]_\alpha [v]_0 + [u]_0 [v]_\alpha$$

$\because \|u\|_{0,\alpha} = [u]_0 + [u]_\alpha$. $\|v\|_{0,\alpha} = [v]_0 + [v]_\alpha$. ✓

Pro 2. (Interpolating inequality)

$$[D^2u]_\alpha \le C\left[[f]_\alpha + |u|_{0;\Omega}\right] \rightarrow \text{Maximum} \quad |f|, |\varphi|$$

$$[D^2u]_\alpha \le C\left[[f]_\alpha + |u|_2\right] \qquad [u]_2 \le \varepsilon [D^2u]_\alpha + C_\varepsilon |u|_0$$

$$\Rightarrow [D^2u]_\alpha \le C\left[[f]_\alpha + |u|_0\right] \quad |u|_{2}$$

Let $\Omega$ is a bounded domain, $u \in C^{2,\alpha}(\bar\Omega)$ $(0<\alpha\le 1)$ For $\forall \varepsilon > 0$.

$$\begin{cases} [u]_2 \le \varepsilon [u]_{2,\alpha} + C_\varepsilon |u|_0 & ① \\ [u]_1 \le \varepsilon [u]_{2,\alpha} + C_\varepsilon |u|_0. & C_\varepsilon = C_\varepsilon(\varepsilon, n, \alpha, \Omega) \end{cases}$$

proof: ①. If not. For every $N>0$. $\exists u_N$ s.t.

$$[u_N]_2 > \varepsilon [u_N]_{2,\alpha} + N|u_N|_0 \qquad \|u_N\|_2 > 0$$

$$\Rightarrow \varepsilon \frac{[u_N]_{2,\alpha}}{\|u_N\|_2} + N \frac{|u_N|_0}{\|u_N\|_2} < 1 \qquad V_n = \frac{u_N}{\|u_N\|_2} \qquad \|V_n\|_2 = 1.$$

$$\Rightarrow \varepsilon [v_N]_{2,\alpha} + N[v_N]_0 < 1.$$

$$\Rightarrow [V_N]_{2,\alpha} < \frac{1}{\varepsilon}, \quad [V_N]_0 < \frac{1}{N}$$

Ascoli-Arzelà:

$$\because \|V_N\|_{2,\alpha} < 1 + \frac{1}{\varepsilon}. \qquad \begin{cases} D^2 V_{N_k} \to D^2 V. & V_{N_k} \to V \text{ in } C^2(\bar\Omega) \\ DV_{N_k} \to DV & \Rightarrow V = 0. \quad \|V\|_2 = 1. \quad \text{contradictory} \\ V_{N_k} \to V. & \end{cases}$$

Pro 3. 

Theorem 1. $\Omega$ has cone property. $V$'s height is $h$. 

$\Rightarrow \ \forall \ 0<\varepsilon\le h$

$$[u]_2 \le \varepsilon^\alpha [u]_{2,\alpha} + \frac{C}{\varepsilon^2}|u|_0$$

$$[u]_1 \le \varepsilon^{1+\alpha}[u]_{2,\alpha} + \frac{C}{\varepsilon}|u|_0 \qquad C = C(\underline{n,\theta,\alpha})$$

Proof: $h=1$. $u\in C^{2,\alpha}(V_1)$ $\overset{\text{Pro 3}}{\Longrightarrow}$ $[u]_{2;V_1} \le [u]_{2,\alpha;V_1} + C|u|_0$.

$u\in C^{2,\alpha}(V_\varepsilon)$.

$y=\frac{x}{\varepsilon}$. $\tilde{u}(y) = u(x) = u(\varepsilon y)$ $\quad \tilde{u}(y)\in C^{2,\alpha}(V_1)$.

$$\Rightarrow [\tilde{u}]_{2;V_1} \le [\tilde{u}]_{2,\alpha;V_1} + C|u|_{0;V_1}$$

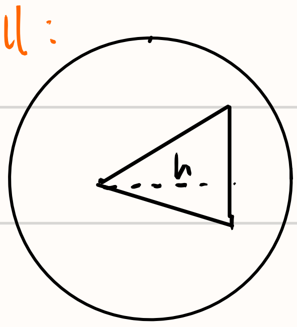
$$\Rightarrow \varepsilon^2[\tilde{u}]_{2;V_\varepsilon} \le [\tilde{u}]_{2,\alpha;V_\varepsilon}\varepsilon^\alpha + C|u|_{0;V_\varepsilon}$$

$$\Rightarrow [\tilde{u}]_{2;V_\varepsilon} \le \varepsilon^\alpha [u]_{2,\alpha;V_\varepsilon} + C|u|_{0;V_\varepsilon}.$$

$$[u]_{2;\Omega} \le \varepsilon^\alpha [u]_{2,\alpha;\Omega} + \frac{C}{\varepsilon^2}|u|_{0;\Omega}.$$

$$[u]_{1;\Omega} \le \varepsilon^{\alpha+1}[u]_{2,\alpha;\Omega} + \frac{C}{\varepsilon}|u|_{0;\Omega}.$$

Ball:



$h=R$.

$$\boxed{\begin{aligned}[u]_{2;\Omega} &\le R^\alpha [u]_{2,\alpha;\Omega} + \frac{C}{R^2}|u|_{0;\Omega}\\ [u]_{2;\Omega} &\le R^{\alpha+1}[u]_{2,\alpha;\Omega} + \frac{C}{R}|u|_{0;\Omega}\end{aligned}}$$

$c(n,\alpha,\theta)$

## §2. Mollifier

Def 1. $u\in L_{loc}(\mathbb{R}^n)$. $\rho$ Mollifier

$$\tilde{u}(x,\tau) \overset{\text{def}}{=} \frac{1}{\varepsilon^n}\int_{B_x(\varepsilon)} u(y)\,\rho\left(\frac{x-y}{\varepsilon}\right)dy.$$

$$\overset{x-y=\varepsilon z}{=\!=} \int_{B_1} u(x-\varepsilon z)\,\rho(z)\,dz \,.$$

Pro 1. Let $u \in C(\mathbb{R}^n)$ whenever $\tau \to 0^+$. $\tilde{u} \rightrightarrows u$ local uniformly.

$\tilde{u} \to u$ $L^p$ $(1 \le p < \infty)$.

$$\sup|\tilde{u}| \le \sup_{B_\tau(x)} |u| \,.$$

$$|D^k \tilde{u}(x,\tau)| \le C(n,k,\rho)\, \tau^{-k} \sup_{B_\tau(x)} |u| \,.$$

$$D^k = D_x^\alpha D_\tau^\beta \,. \qquad |\alpha| + |\beta| = |k|$$

$$\tilde{u}(x,\tau) \overset{-u(x)}{=} \int_{B_1} u(x-\varepsilon z)\,\rho(z)\,dz - \int_{B_1} \rho(z)\,u(x)\,dz \,.$$

$$= \int_{B_1} (u(x-\varepsilon z) - u(x))\,\rho(z)\,dz$$

$$|\tilde{u}(x,\tau) - u(x)| \le \int_{B_1} \left(|u(x-\varepsilon z) - u(x)|\right) \rho(z)\,dz \,.$$

$$\le \max_{z\in B_1} |u(x-\varepsilon z) - u(x)| \to 0 \quad (\varepsilon \to 0) \,.$$

$$|\tilde{u}(x,\tau)| \le \int_{B_1} |u(x-\varepsilon z)|\,\rho(z)\,dz \le \max_{z\in B_1} |u(x-\varepsilon z)| = \max_{B_\tau(x)} |u(x)| \,.$$

$$\tilde{u}(x,\tau) = \frac{1}{\tau^n} \int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\tau}\right) u(y)\,dy \,.$$

$$D_{x_k} \tilde{u}(x,\tau) = \frac{1}{\tau^{n+1}} \int_{B_{x}(\tau)} \rho_k\left(\frac{x-y}{\tau}\right) u(y)\,dy$$

$$D_\tau \tilde{u}(x,\tau) = \frac{-n}{\tau^{n+1}} \int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\tau}\right) u(y)\,dy + \frac{1}{\tau^n} \int_{\mathbb{R}^n} \nabla\rho\left(\frac{x-y}{\tau}\right) \cdot \left(-\frac{x-y}{\tau^2}\right) u(y)\,dy$$

$$= \frac{1}{\tau^{n+1}} \int_{\mathbb{R}^n} \left[\rho\left(\frac{x-y}{\tau}\right) - \nabla\rho\left(\frac{x-y}{\tau}\right)\left(\frac{x-y}{\tau}\right)\right] u(y)\,dy$$

$$= \frac{1}{\tau^{n+1}} \int_{\mathbb{R}^n} \rho_1\left(\frac{x-y}{\tau}\right) u(y)\,dy \qquad \rho_1 \in C_0^\infty(\mathbb{R}^n) \,.$$

Induct: $D^k \tilde{u}(x,\tau) = \tau^{-n-k} \int_{\mathbb{R}^n} \rho_k\left(\frac{x-y}{\tau}\right) u(y)\,dy$.

$$\Rightarrow |D^k \tilde{u}(x,\tau)| \le C\tau^{-k} \sup_{B_\tau(x)} |u| \,.$$

Pro 2. Let $u \in C_{loc}^\infty(\mathbb{R}^n)$.

$\Rightarrow$ ① $|\tilde{u}(x,\tau) - u(x)| \le \tau^\alpha H_x^\alpha[u; B_\tau(x)]$.

② $|D^k \tilde{u}(x,\tau)| \le C(n,\alpha,k,\rho)\, H_x^\alpha[u; B_\tau(x)]$

proof: $\tilde{u}(x,\tau) - u(x) = \int_{\mathbb{R}^n} \rho(y)\, u(x-\tau y) - \rho(y)\, u(x)\, dy$.

$$= \int_{B_1} \rho(y)[u(x-\tau y) - u(x)]\, dy$$

$\Rightarrow |\tilde{u}(x,\tau) - u(x)| \le |\tau y|^\alpha H_x^\alpha[u; B_\tau(x)] \le \tau^\alpha H_x^\alpha[u; B_\tau(x)]$

$D^k = D_x^{k_1} D_\tau^{k_2}$, $|k_1| + |k_2| = |k|$

$k_2 \neq 0$. $D^k \tilde{u}(x,\tau) = D^k \tilde{u}(x,\tau) - D^k u(x) = \tau^{-n-k} \int_{\mathbb{R}^n} \rho_k\left(\frac{x-y}{\tau}\right) u(y) - \underbrace{\int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\tau}\right) u(x)\, dy}^{\tau^{-n-k}}$

$$= \tau^{-k} \int_{\mathbb{R}^n} \rho_k(z)\,[u(x-\tau z) - u(x)]\, dz.$$

$\Rightarrow |D^k \tilde{u}(x,\tau)| \le C(n,\alpha,k,\rho)\, H_x^\alpha[u; B_\tau(x)]$

$k_2 = 0$ $\quad D^k \tilde{u}(x,\tau) = \tau^{-n} \int_{\mathbb{R}^n} D^k\left[\rho\left(\frac{x-y}{\tau}\right)\right] u(y)\, dy$

$$= \tau^{-n} \int_{\mathbb{R}^n} D^k\left[\rho\left(\frac{x-y}{\tau}\right)\right] (u(y) - u(x))\, dy + \tau^{-n} \int_{\mathbb{R}^n} D^k\left[\rho\left(\frac{x-y}{\tau}\right)\right] u(x)\, dy$$

$$= \tau^{-n} \int_{\mathbb{R}^n} D^k\left[\rho\left(\frac{x-y}{\tau}\right)\right] (u(y) - u(x))\, dy + u(x)\tau^{-n}(-1)^{|k|} \int_{\mathbb{R}^n} D_y^k\left[\rho\left(\frac{x-y}{\tau}\right)\right] dy$$

$$= \tau^{-n} \int_{\mathbb{R}^n} D^k\left[\rho\left(\frac{x-y}{\tau}\right)\right] (u(y) - u(x))\, dy$$

$\Rightarrow |D^k \tilde{u}(x,\tau)| \le C(n,\alpha,k,\rho)\, H_x^\alpha[u; B_\tau(x)]$

Pro 3. Let $u \in C(\mathbb{R}^n)$ If for $0 < \alpha \le 1$, $R > 0$

$$\sup_{y \in B_R(x),\, 0<\tau<R} \tau^{1-\alpha} |D\tilde{u}(y,\tau)| < \infty. \quad \longrightarrow D_x, D_\tau$$

$\Rightarrow H_x^\alpha[u; B_R(x)] < \infty$ i.e. $H_x^\alpha[u; B_R(x)] \le C(n,\alpha,\rho) \sup_{y \in B_R(x),\, 0<\tau<R} \tau^{1-\alpha} |D\tilde{u}(y,\tau)|$

proof: for $|x-y| < R$. $0 < \tau \le R$.

$$|u(x) - u(y)| \le |\tilde{u}(x,\tau) - u(x)| + |\tilde{u}(x,\tau) - \tilde{u}(y,\tau)| + |\tilde{u}(y,\tau) - u(y)|.$$

note that when $0 < \tau \le R$

$$|\tilde{u}(x,\tau)-u(x)| = \left|\int_{\mathbb{R}^n} \rho(y)u(x-\tau y) - \rho(y)u(x)\,dy\right|$$

$$= |\tilde{u}(x,\tau) - \tilde{u}(x,0)|$$

$$= \left|\int_0^\tau D_\tau \tilde{u}(x,y)\,dy\right| \xlongequal{y=\tau s} \tau\left|\int_0^1 D_\tau \tilde{u}(x,\tau s)\,ds\right|.$$

$$\leq \tau^\alpha \int_0^1 \frac{(\tau y)^{1-\alpha}|D_\tau \tilde{u}(x,y\tau)|}{y^{1-\alpha}}\,dy$$

$$\leq \sup_{0<\tau<R} (y\tau)^{1-\alpha}|D_\tau\tilde{u}(x,y\tau)| \cdot \tau^\alpha \int_0^1 \frac{1}{y^{1-\alpha}}\,dy = \frac{\tau^\alpha}{\alpha}\sup_{0<\tau<R}\tau^{1-\alpha}|D_\tau\tilde{u}(x,\tau)|$$

$$\Rightarrow |u(x)-u(y)| \leq \frac{2}{\alpha}\tau^\alpha \sup_{\substack{0<\tau<R\\ z\in B_R(x)}} \tau^{1-\alpha}|D\tilde{u}(z,\tau)| + |D_x\tilde{u}(x^*,\tau)||x-y|.$$

Let $\tau = |x-y|$ $\Rightarrow y\in B_r(x)$. $\dfrac{|u(x)-u(y)|}{|x-y|^\alpha} \leq \left(\dfrac{2}{\alpha}+1\right)\sup\limits_{\substack{0<\tau<R\\ z\in B_R(x)}} \tau^{1-\alpha}|D\tilde{u}(z,\tau)|$

Conclusion:

① If $u\in C^{0,\alpha}(\mathbb{R}^n)$. $\Rightarrow \exists\, C=C(n,\alpha,\rho)$. s.t

$$\frac{1}{C}[u]_\alpha \leq \sup_{\tau>0,\,x\in\mathbb{R}^n}\tau^{1-\alpha}|D\tilde{u}(x,\tau)| \leq C[u]_\alpha.$$

② $u\in C^{|\beta|+1,\alpha}$

$\Rightarrow \exists\, C(n,\alpha,\rho)$

$$[D^{\beta+1}u]_\alpha \leq \sup_{\tau>0}[DD_x^\beta\tilde{u}(x,\tau)]_\alpha^X \leq C[D^{|\beta+1|}u]_\alpha.$$

Proof: $\dfrac{|D_x^{\beta+1}\tilde{u}(x,\lambda) - D_x^{\beta+1}\tilde{u}(y,\lambda)|}{|x-y|^\alpha} \leq \sup\limits_{\tau>0}[D_x^{\beta+1}\tilde{u}]_\alpha^X$

Let $\lambda\to 0$. $\Rightarrow [D_x^{\beta+1}u]_\alpha \leq \sup\limits_{\tau>0}[D_x^{\beta+1}\tilde{u}]_\alpha^X$

令 $y=x+h$. $A_h u = u(x+h)$. $|\underbrace{DD_x^\beta\tilde{u}(x,\tau) - DD_x^\beta\tilde{u}(y,\tau)}|$

$$= |DD_x^\beta\widetilde{(u-A_hu)}(x,\tau)| = |DD_x^\beta(u-A_hu)(x,\tau)| \underset{\alpha=1}{\overset{k=1}{\leq}} CH_x^1[D^\beta(u-A_hu)(x,\tau)]$$

$$\leq C[D^{\beta+1}(u-A_hu)]_0 \quad \Rightarrow \sup_{\tau>0}[DD_x^\beta\tilde{u}(x,\tau)]_\alpha^X \leq C[D^{\beta+1}u]_\alpha.$$