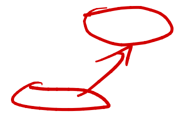
Conservation laws and shock waves.

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u + pI) = 0, \\ \partial_t(\rho E) + \nabla\cdot(\rho E u + pu) = 0, \end{cases} \quad p = p(\rho, e), \quad \rho E = \frac{1}{2}\rho|u|^2 + \rho e.$$

Euler

$\partial_t v + \nabla\cdot(f(v)) = 0$, $\rightarrow$ conservation law. (hyperbolic).

$v = \underbrace{(\rho,\ \rho u,\ \rho E)}_{1 \quad n \quad 1}$, $\quad (\rho u,\ \rho u\otimes u + pI,\ \rho E u + pu) = f(v)$

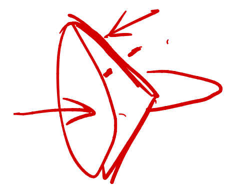


NS. $\quad \partial_t v + \nabla\cdot(f(v)) = \nabla\cdot(B(u,\varepsilon)\nabla u)$. $\quad n+2$, NS, $\varepsilon = 0$, $(2\mu+\lambda)$, $n+2$ $\rightarrow$ toy model $\rightarrow$

Schrödinger / incompress Euler/NS $\Longleftrightarrow$ compressible Euler/NS

$\downarrow$

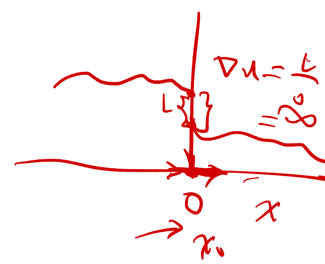
shock



- $|\nabla u| = \infty$

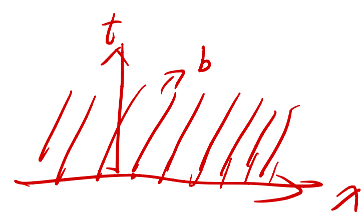
  $x = x_0 = f(x_0)$

- RH condition
- entropy condition



method

✓ characteristic curve



$u_t + b u_x = 0$

transport equation

Evans chapter 3.

conservation law

Toy model:

$\partial_t u + u\cdot\nabla u + \nabla p = 0$. Incompressible Euler; $\nabla\times$

$\|u(t)\|_{H^s} \to \infty \Leftrightarrow \int_0^{T^*} \|w\|_\infty = \infty$, $\quad w = \nabla\times u$

$t \to T^*$, $\quad L^1 L^\infty(0, T^*)$

$\partial_t w + u\cdot\nabla w = w\cdot\nabla u$. vorticity equation. $\rightarrow$ axisymmetric

3, ε

wTw $\rightarrow$ T: $\Delta^{-1}\nabla\nabla$

①

$\rightarrow \boxed{T = \frac{c}{\partial\theta} H}$

$\partial_t w = w H w \rightarrow$ 1D toy model of vorticity equation.

$w = w_0,$

$$w = \frac{w_0}{(1 + t H(w_0))^2 + t^2 w_0^2} \longleftrightarrow \textcircled{w_0} = \sin x.$$

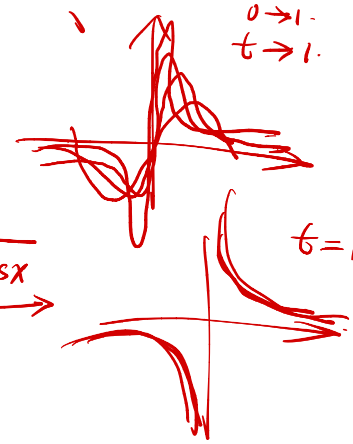
$\downarrow < 0$

$t \rightarrow \frac{1}{Hw_0}$

$$w = \frac{\sin x}{1 + t^2 - 2t\cos x}$$

$\downarrow t \quad 0 \rightarrow 1$

$$\frac{\sin x}{2 - 2\cos x} \rightarrow x = 0 \qquad \| w(1) \|_{L^\infty} = \infty$$



Elgindi 2021 $\quad C^{1,\alpha} \rightarrow$ blow up

C^∞

$\Updownarrow$

C^2

---

1 d conservation law.

scalar

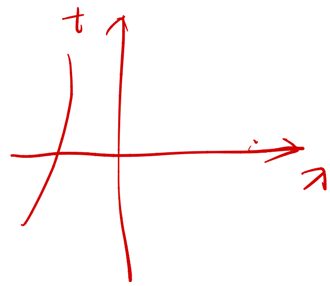
$\partial_t u + \nabla \cdot (f(u))$

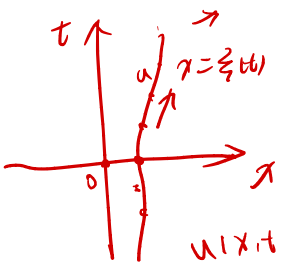
$\downarrow \qquad \downarrow$

$u \qquad \partial_x f(u) =$

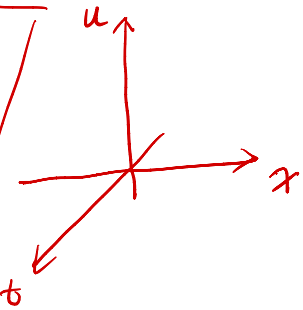
$\partial_t u + \partial_x f(u) = 0$

$\Rightarrow \quad u_t + (f(u))_x = 0$





$u(x,t) \equiv u(x_0, 0)$



① $x = \xi(t),$ $\qquad \frac{d}{dt} u(\xi(t), t) \equiv 0 \Rightarrow \xi'(t)\, u_x(\xi(t), t) + u_t(\xi(t), t) = 0.$

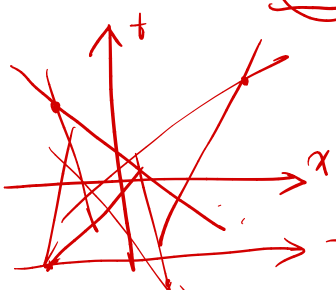
$\partial_t u + f(u))_x \qquad (u_t + \xi' u_x)(\xi(t), t) = 0.$

$u \equiv C$

$$\boxed{\xi' = f'(u) = \lambda(u)}$$

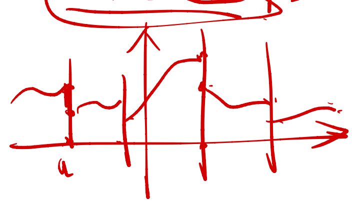
$\rightarrow x = \xi(t)$ is a line. $\quad \xi'(t) = f'(u)(\xi(t), t) = \lambda(u)(\xi(t), t) \equiv \lambda(u)(\xi(0), 0)$

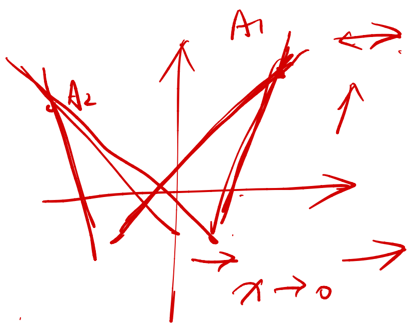
u



piecewise smooth solution

(in distributional sense)





$x \to 0$ (compression wave). shock. $f''(u) \geq 0$

A_1: $1 \to 2$

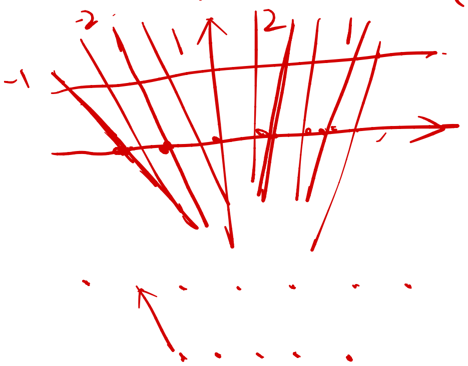
A_2: $-2 \to -1$

$(\xi')_x \geq 0$

$\xi' = f'(u) \to f''(u)(\xi(t),t) \geq 0$

$x \to 0$

(expansion wave) (rarefaction wave formation)



$f''(u) < 0$

$f''(u) \leq 0$

$f''(u) \geq 0$

f is convex function

$f(u) = \frac{1}{3}u^3$

$f''(u) < 0$

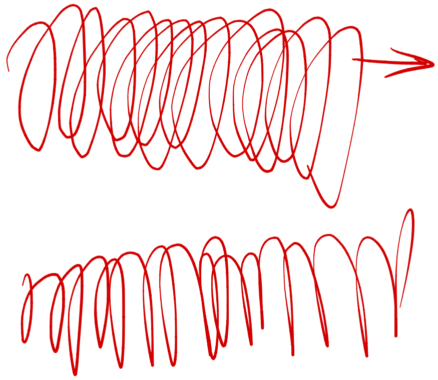
$f''(u) = u$

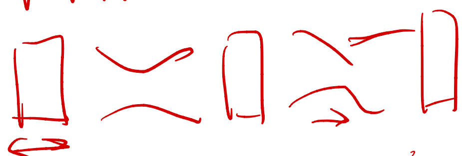
$f''(u) > 0$

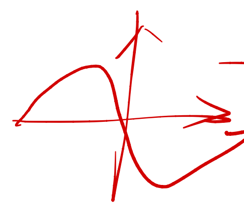
$u > 0$

≤ 0

$u = 0$







convex

compression

expansion

convex concave

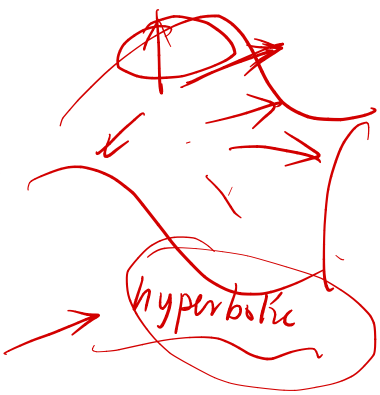
$f''(u) > 0$ $f''(u) < 0$

conservation system

$f''(u) > 0$

$f''(u) < 0$

$f''(u)$



Hopf equation: $u_t + u u_x = 0$ critical

(inviscid equation)

Burgers

$f(u) = \frac{1}{2}|u|^2$ $f''(u) = 1 \to$ convex

$(\rho u)_t + \nabla \cdot (\rho u \otimes u)$

$u_t + \nabla u^2 = 0$

Some examples of shock waves and rarefaction waves:

RH condition: $\to$ motion of shock wave.

entropy condition $\to$ uniqueness solution.

(inviscid Burgers).

We consider Hopf equation:

$$\begin{cases} u_t + u u_x = 0, \\ u(0) = u_0. \end{cases}$$

$\xi'(t) = 1 - x$

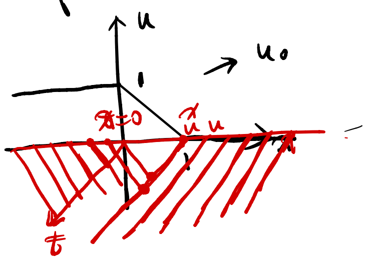
$(\frac{1}{2}u)' = u = 1$

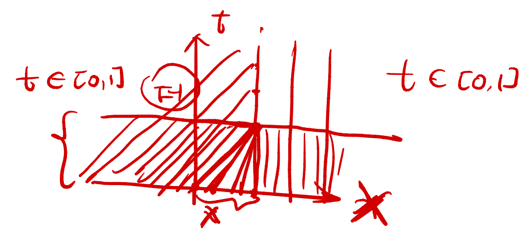
$\xi'(t) = f'(u) = u = 0 \, 1$

t

① $u_0 = \begin{cases} 1, & x \le 0, \\ 1-x, & x \in [0,1], \\ 0 & x \ge 1. \end{cases}$

$u_0 \equiv 0$





$$x = \begin{cases} X + t, & x \le 0. \\ X + (1-X)t, & x \in [0,1] \\ X + \cancel{t}, & x > 0 \end{cases}$$

$(X, t) \to x$

$(x, t) \to X$

$$X = \begin{cases} x - t, & x \le t, t \in [0,1] \\ \frac{x-t}{1-t}, & t \le x \le 1, t \in [0,1] \\ x & x \ge 1, t \in [0,1] \end{cases}$$

$x = \cancel{X} + (1 - \cancel{X})t$ $\quad t = 0, \quad t = 1$

$t = 1 \quad t = 1$

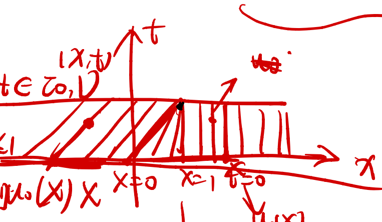
$X + (1 - X) = 1.$

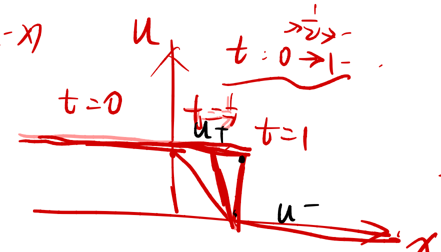
$$\boxed{\begin{array}{ccc} & (x,t) \to X & \\ & & \downarrow \\ u(x,t) & \leftarrow & u_0(X) \end{array}}$$

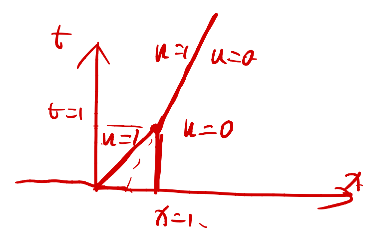
$$u(x,t) = u_0(X) = \begin{cases} 1, & x \le t, t \in [0,1] \\ \frac{1-x}{1-t}, & t \le x \le 1, t \in [0,1] \\ 0, & x \ge 1, t \in [0,1] \end{cases}$$

$t = 1$

$2(1-x)$







② Rankine-Hugoniot condition:

RH

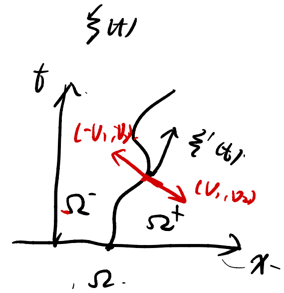
$$\xi'(t) = \frac{f(u_+) - f(u_-)}{u^+ - u^-} = \frac{\frac{1}{2}u_+^2 - \frac{1}{2}u_-^2}{u^+ - u^-} = \frac{u_+ + u_-}{2}$$

$f(u) = \frac{1}{2}u^2$

Theorem. suppose (u_+, u_-), (jump condition), u satisfies conservation law: $u_t + f(u)_x = 0$,

then $(u_+, u_-) \to \xi'(t)$ satisfies: $\xi'(t) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}$.

Proof: $0=\int_0^{+\infty}\int_{-\infty}^{+\infty}(uv_t+f(u)v_x)\,dx\,dt$ $\qquad \dfrac{v\in C_c^\infty(\mathbb{R}\times[0,\infty))}{v(x,0)=0}$



$$=\int_{\Omega^-}+\int_{\Omega^+}(uv_t+f(u)v_x)\,dx\,dt$$

$$=\int_{\Omega^-}+\int_{\Omega^+}(f(u),u)\cdot\nabla_{x,t}v\,dx\,dt$$

$$=\int_{\Omega^-}+\int_{\Omega^+}\underbrace{\nabla_{x,t}(f(u),u)}_{(u_t+f(u)_x)}v\,dx\,dt+\int_{\partial\Omega^-}v(f(u),u)\cdot \underset{(v_1,v_2)}{n}\,dx\,dt$$

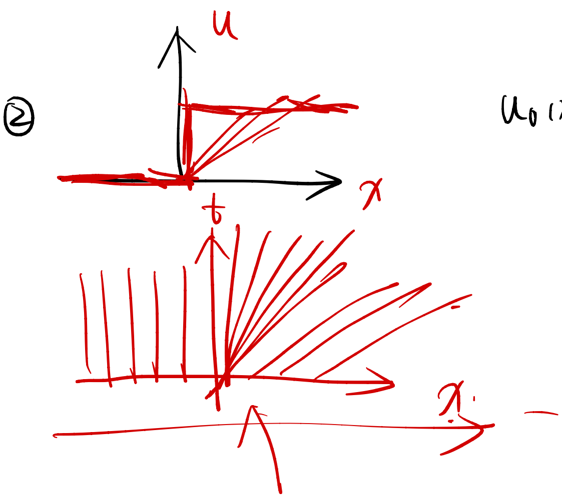
$$+\int_{\partial\Omega^+}v(f(u),u)\cdot\underset{(-v_1,-v_2)}{n}\,dx\,dt$$

$$=\int_\tau v(f(u^+),u^+)\cdot(v_1,v_2)+\int v(f(u^-),u^-)(-v_1,-v_2)$$

$$=\int_\tau v(f(u)^+-f(u^-),\ u^+-u^-)\cdot(v_1,v_2)$$

$$\frac{f(u^+)-f(u^-)}{u^+-u^-}=-\frac{v_2}{v_1}=\xi'(t)$$

② $u_0(x)=\begin{cases}0, & x\le 0\\ 1, & x\ge 0.\end{cases}$



entropy condition

$u^+>\xi'(t)>u^-$

0 $\qquad$ 1