# Lecture 1. Scalar conservation law.

First we'd like to consider the following scalar conservation law:

$$u_t + (f(u))_x = 0, \quad x, t \in \mathbb{R} \times [0, \infty) \qquad (1)$$

And we have following theorem for the jump discontinuity:

**Theorem 1.** Suppose $u$ is a piecewise smooth function with a jump discontinuity $(u_+, u_-)$ along a trajectory $\tau$: $x = \xi(t)$, where

$$(u_+, u_-)(t) =: (u(\xi(t)+, t), u(\xi(t)-, t)).$$

Then $u$ satisfies the scalar conservation law in weak sense if and only if $(u_+, u_-)$ satifies the following

Rankine-Hugoniot condition: $\quad f(u_+) - f(u_-) = \xi'(u_+ - u_-). \qquad$ (RH)

Remark: Particular, the speed of motion of the jump $\sigma =: \xi' = \dfrac{f(u_+) - f(u_-)}{u_+ - u_-}$.

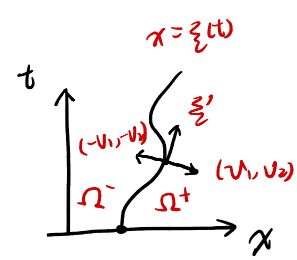
Proof:

$\Rightarrow$ suppose $u$ satisfies the weak scalar conservation law, i.e.

$$\int_0^\infty \int_{-\infty}^{+\infty} (u v_t + f(u) v_x)\, dx\, dt + \int_{-\infty}^{+\infty} u(0) v(0)\, dx = 0, \quad \forall v \in D(\mathbb{R} \times [0, \infty))$$

Now we choose proper test function to verify the RH condition. Indeed, we set $v(x, 0) = 0$, $\forall x \in \mathbb{R}$. The jump path $x = \xi(t)$ break the domain into two parts:

$$\Omega^- = \{(x,t) \in \mathbb{R} \times (0, \infty) \mid x < \xi(t)\},$$
$$\Omega^+ = \{(x,t) \in \mathbb{R} \times (0, \infty) \mid x > \xi(t)\}.$$




Then it yields that

$$0 = \int_0^\infty \int_{-\infty}^{+\infty} (u v_t + f(u) v_x)\, dx\, dt$$

$$= \int_{\Omega^-} (u v_t + f(u) v_x)\, dx\, dt + \int_{\Omega^+} (u v_t + f(u) v_x)\, dx\, dt$$

$$= \int_{\Omega^-} (f(u), u) \cdot \nabla_{x,t} v\, dx\, dt + \int_{\Omega^+} (f(u), u) \cdot \nabla_{x,t} v\, dx\, dt$$

$$= -\int_{\Omega^-} \nabla_{x,t}\,(f(u), u)\, v\, dx\, dt + \int_{\partial\Omega^-} v\,(f(u), u)\cdot n\, dx\, dt$$

$$-\int_{\Omega^+} \nabla_{x,t}\,(f(u), u)\, v\, dx\, dt + \int_{\partial\Omega^+} v\,(f(u), u)\cdot n\, dx\, dt$$

since $v \equiv 0$ for $t=0$.

$$= -\int_{\Omega^-} (u_t + (f(u))_x)\, v\, dx\, dt + \int_{\tau} v\,(f(u^-), u^-)\cdot(\nu_1, \nu_2)\, dx\, dt$$

vanish since $u$ is smooth in $\Omega^+/\Omega^-$

$$-\int_{\Omega^+} (u_t + (f(u))_x)\, v\, dx\, dt + \int_{\tau} v\,(f(u^+), u^+)\,(-\nu_1, -\nu_2)\, dx\, dt$$

$$= -\int_{\tau} v\,(f(u^+) - f(u^-),\, u^+ - u^-)\,(\nu_1, \nu_2)\, dx\, dt.$$

Since $v$ is arbitray in $D(\mathbb{R}\times[0,\infty))$ with $v(0)\equiv 0$. Then it comes:

$$\frac{f(u^+)-f(u^-)}{u^+-u^-} = \frac{-\nu_2}{\nu_1} = \xi'.$$

and thus the necessity is finished.

$\Longleftarrow$: suppose (RH) holds, we show that $u$ satifies (1) in weak sense indeed.

# Method of characteristic lines

In this section, we'd like to consider the following scalar conservation law:

$$u_t + (f(u))_x = 0.$$

The characteristic analysis play a important role when we study the hyperbolic system. Indeed, we consider the characteristic curve $x = \xi(t)$ such that $u(x,t) = u(\xi(t),t)$ stay constant.

Then it comes that

$$\frac{d}{dt} u(\xi(t),t) = (u_t + \xi' u_x)(\xi(t),t) \equiv 0 \Rightarrow \frac{d}{dt}\xi(t) = f'(u)(\xi(t),t).$$

Consequently, we'd like call $\lambda(u) = f'(u)$ as the characteristic speed. Particularly, we will find $\lambda(u)$ is a constant along $x = \xi(t)$ since it is a function of $u$:

$$\lambda(u)(\xi(t),t) = f'(u)(\xi(t),t) = f'(u)(\xi(0),0) = \lambda(u)(\xi(0),0).$$

consequently, the characteristic curve is a line for any initial point $X = \xi(t)$, then we can solve out value of any point $(x,t)$ as long as they stay reasonable:

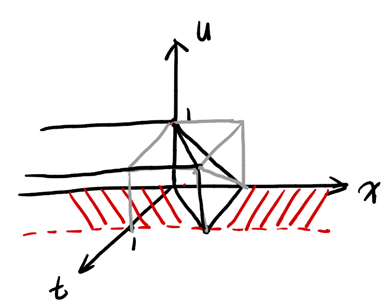
$$u(x,t) = u(x - \lambda(u)(x,t)t,\ 0) = u_0(x - \lambda(u)(x,t)t).$$

Following we consider the most important equation of scalar conservation law:

Hopf (inviscid Burgers) equation: $\quad u_t + (\frac{1}{2}u^2)_x = 0.$

Example 1: we consider the following initial data:

$$u_0(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & x \in [0,1], \\ 0, & x > 1. \end{cases}$$

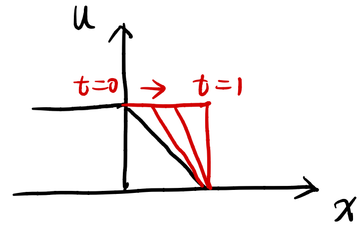



The characteristic speed $(\lambda(u) = f'(u) = (\frac{1}{2}u^2)' = u)$ $\lambda(x,0) = u_0(x)$, and so the characteristic is as the figure. We should notice the lines may intersect after $t \geq 1$, so we'd like to analyse the behavior in $t \in [0,1]$ first. Then given initial position, the lines goes as

$$x = \begin{cases} X + t, & X \in [-\infty, 0) \\ X + (1-X)t, & X \in [0,1] \\ X, & X \in [1,\infty) \end{cases} \Rightarrow X = \begin{cases} x - t, & x \leq t, \quad t \in [0,1] \\ \frac{x-t}{1-t} & t \leq x \leq 1, \quad t \in [0,1] \\ x, & x \geq 1, \quad t \in [0,1] \end{cases}$$

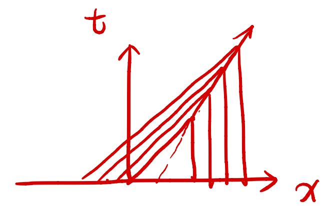
consequently, we have that:

$$u(x,t) = u(X,0) = u_0(X) = \begin{cases} 1, & x \le t,\ t \in [0,1] \\ \frac{1-x}{1-t}, & t \le x \le 1,\ t \in [0,1] \\ 0, & x \ge 1,\ t \in [0,1]. \end{cases}$$




From Example 1. we see that it forms a shock $u(x,1) = \begin{cases} 1, & x \le 1, \\ 0, & x > 1. \end{cases}$, to analyse the behavior of the solution (particularly the jump discontinuity), we introduce the following theorems:

Example 2. Consider the initial data $u_0 = \begin{cases} 1, & x \le 1, \\ 0, & x \ge 1. \end{cases}$ we analyse the motion of the jump discontinuity. It follows from RH condition that:



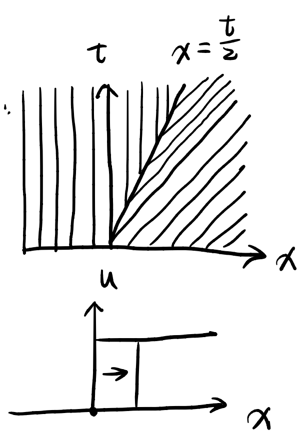

$$\sigma = \xi' = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{u^+ + u^-}{2} = \frac{1}{2}.$$

That is say, the jump moves forward at speed $\frac{1}{2}$.

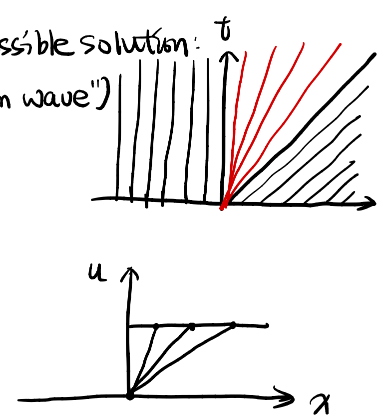
Now we introduce another kind of phenomena, which is called rarefaction wave.

Example 3. consider the initial data $u_0 = \begin{cases} 0, & x \le 0, \\ 1, & x \ge 0. \end{cases}$

One possible solution:
("shock wave")




another possible solution:
("rarefaction wave")

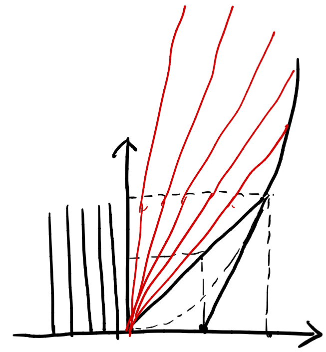



The first case does not happen in nature, which is excluded by following entropy condition:

$$f'(u^-) > \sigma > f'(u^+).$$

we say a weak solution is admissible if the discontinuity satisfies the RH condition and the entropy condition both.

Example 4. We consider the initial data $u_0 = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$

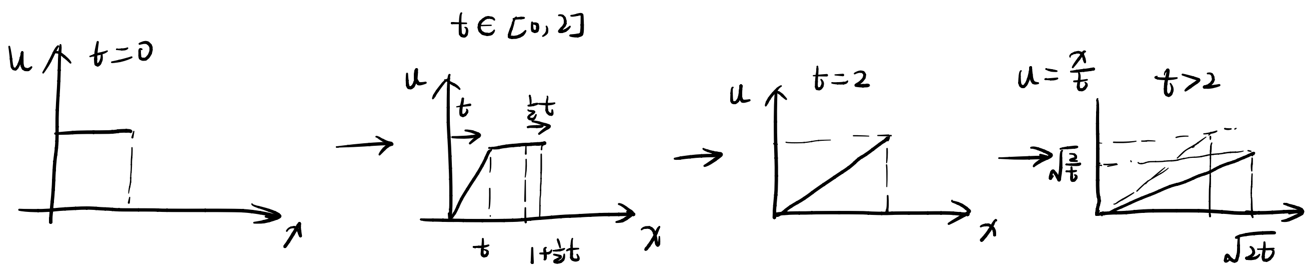


from RH condition we have that:

$$\xi'(t) = \frac{f(u^+)(t) - f(u^-)(t)}{u^+(t) - u^-(t)} = \frac{\frac{1}{2}(\xi(t)/t)^2}{(\xi(t)/t)} = \frac{\xi(t)}{2t}, \quad t > 2.$$

solve the ODE we have: $\xi(t) = \sqrt{2t}$, $t > 2$.

The phenomeno is as:




## Long time behavior

① $\|u\|_{L^\infty} \lesssim t^{-\frac{1}{2}}$.

② $\|u - N\|_{L^1} \lesssim t^{-\frac{1}{2}}$.

## Hyperbolic conservation law:

We consider the Euler equation:

$$\begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} + \nabla \begin{pmatrix} \rho u \\ \rho u \otimes u + pI \\ \rho E u + pu \end{pmatrix} = 0.$$

Now $p$ can be obtain as $p = p(\rho, u, E)$.

The general conservation law is written as:

$$u_t + \nabla \cdot \mathbb{F}(u) = \nabla \cdot (B(u,\varepsilon) \nabla g(u)). \quad \text{viscous term.}$$