Lecture. Global regularity of the compressible flow. (with vacuum).

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u)=0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla p = \mu\Delta u + (\mu+\lambda)\nabla(\nabla\cdot u), \\ p=\rho^\gamma. \end{cases}$$

$\begin{cases} \text{Kazhikhov-Vaigant model: 2D, } \lambda=\rho^\beta,\ \beta>0. \\ \text{3D, classic model } (\mu, \lambda \text{ are constant}) \end{cases}$

global regularity $\leftarrow$ local regularity Δ (existence of regular solution) $\rho\in C^\alpha$. $\{$ Cho-Kim 03-06, vaccum local existence.

$\nwarrow$ a propri estimate Δ Δ

A propri estimate:

global existence $\leftarrow$ density $\|\rho\|_{L^\infty}\le C_T$ (uniform). Δ

II. BKM blowup criterion: $\lim_{T\to T^*}\|\rho\|_{L^\infty_{(0,T)}}=\infty$, $\|\rho\|_{L^\infty}<\infty$.

density uniform boundedness: $\|\cdot\|_{L^\infty}\le C$.

① De-Giorgi estimate: L^p-bounded $\to$ L^∞-bounded Δ

good: $\|\rho\|_{L^\infty}\le C\|v\|_{L^p}$. CNS $\to$ $\rho=0$ $\to$ hyperbolic.

bad: elliptic / parabolic $\leftarrow$ hyperbolic. Δ

② Gronwall type argument: $f'(t)\le h(f(t))$ Gronwall. $\Rightarrow$ $f(t)\le C_t$.

$\|\rho\|_{L^\infty}$

$\downarrow$ Huang-Li $\begin{cases} \text{12 3D classic} \\ \text{16 2D Kazhikhov} \to \end{cases}$

Zlotnik inequality.

3D classic model: $\|\rho\|_{L^\infty} \leftarrow \|\hat\rho\|_{L^\infty}$ $\boxed{\|\rho\|_{L^\infty_{(\Omega)}}(t)} \le C_0$ $[0,$

$\rho(x,t)$ Euler coordinate. $D_t\rho = -\rho\nabla\cdot u \to \dfrac{F+P}{2\mu+\lambda}$

$\hat\rho(X,t)$, Lagrange coordinate $\partial_t\hat\rho$

$F = (2\mu+\lambda)\nabla\cdot u - P$

x,t

$\rho(x(X,t)) = \hat\rho(X,t)$

$\hat\rho_X(t) < C$ independent of X.

$$D_t\rho = -\rho\,\frac{F+P}{2\mu+\lambda}$$

$$= -\frac{\rho P}{2\mu+\lambda} - \frac{F\rho}{2\mu+\lambda}$$

$\to \hat\rho_X(t) \le C_0$

$\Downarrow$

$\rho(t) \le C_0$

$b' = \partial_t\left(\int_0^t \dfrac{F\rho}{2\mu+\lambda}\right)$

$\hat\rho$

$$y'(t) = g(y)(t) + b'(t), \quad \begin{cases} g(-\infty) = -\infty \\ b(t_1) - b(t_2) \le N_0 + N_1(t_2 - t_1), \end{cases}$$

$y(t)$ is upper bounded,

$$y(t) \le \bar y = \max\{y_0, \hat\xi\} + N_0, \quad \hat\xi = \xi(N_1)$$

2D Kazhikov model: $\Rightarrow \lambda = \rho^\beta \nearrow$, $\beta > 0$.

F:
$(2\mu+\lambda(\rho))\nabla\cdot u - P$

$D_t\rho = -\rho\nabla\cdot u$

$$-\frac{2\mu+\lambda(\rho)}{\rho}(\rho\nabla\cdot u) = -(2\mu+\lambda(\rho))\nabla\cdot u = -F - P$$

$$\frac{(2\mu+\lambda(\rho))}{\rho} D_t\rho = -F - P$$

$\theta'(\rho)$, $D_t\rho \to \theta(\rho)$

$\theta'(\rho) = \rho^{-1} + \rho^{\beta-1}$

$\rho^{\beta-1}$ $\beta > 0$

$D_t\theta(\rho) = -F - P$

$\theta(\rho)$

ρ

$\theta(\rho) \sim \rho^\beta$

$$D_t\theta(\rho) = -F - P$$

$$= \xi$$

$$= -\partial_t(\Delta^{-1}\nabla(\rho u))$$

$$-(\Delta^{-1})\nabla\cdot(\nabla\cdot(\rho u\otimes u)) - P$$

$$D_t\theta = D_t\xi + \boxed{u\cdot\nabla\xi} - (\Delta^{-1})(\nabla\cdot\nabla\cdot(\rho u\otimes u)) - P$$

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) = (2\mu+\lambda)\nabla(\nabla\cdot u) - \mu\nabla\times\nabla\times u - \nabla P$$

$$= \nabla\underbrace{((2\mu+\lambda)(\nabla\cdot u) - P)}_{F} + \mu\nabla\times(\nabla\times u)$$

$\nabla\cdot$ $\quad$ $\nabla\times$ $\quad$ Δ

$$\boxed{\partial_t(\nabla\cdot(\rho u)) + \nabla\cdot(\nabla\cdot(\rho u\otimes u)) = \Delta F}$$

$$\rho D_t u$$

$$u\cdot\nabla\xi - (\Delta^{-1})\nabla\cdot(\nabla\cdot(\rho u\otimes u))$$

②

$$\nabla\cdot(\rho D_t u) = \Delta F$$

$$F = \Delta^{-1}(\nabla\cdot(\rho D_t u))$$

$$= \Delta^{-1}(\nabla\cdot(\partial_t(\rho u) + \nabla(\rho u\otimes u)))$$

$$= \partial_t(\Delta^{-1}\nabla\cdot(\rho u))$$

$$+ (\Delta^{-1})\nabla\cdot(\nabla\cdot(\rho u\otimes u))$$

$$\underbrace{u\cdot\nabla(\Delta^{-1}\nabla\cdot(\rho u))}_{①} = -[u^i, R_iR_j](\rho u^j)$$

m (commutator).

$$D_t\theta = -P + D_t\xi - [u^i, R_iR_j](\rho u^j)$$

$b(t)$

$$D_t\theta = -P + D_t\left(\xi - \int_0^t m\right)$$

$\rho \Leftrightarrow \theta$

$$P = P(\rho) = P(\theta)$$

$$D_t\theta = -P(\theta) + D_t\left(\xi - \int_0^t m\right)$$

$$y'(t) = g(y)(t) + b'(t) \leftarrow \text{Zlotnik inequality}$$

3D classic model (Li, Huang-Li-Xin)

$A_1(t) = \sup(\sigma \|\nabla u\|_{L^2}^2) + \int_0^T \sigma \|\rho^{\frac12} \dot u\|_{L^2}^2.$

$A_2(t) = \sup(\sigma^3 \|\rho^{\frac12} \dot u\|_{L^2}^2) + \int_0^T \sigma^3 \|\nabla \dot u\|_{L^2}^2$

$A_3(t) = \int \rho |u|^3 = \sup \|\rho^{\frac13} u\|_{L^3}^3.$

$\sigma(t) = \begin{cases} t, & t \le 1 \\ 1, & t \ge 1. \end{cases}$

$C_0 = \int \frac12 \rho_0 |u_0|^2 + P(\rho_0).$

if $C_0 \le \varepsilon$, $A_1(T) + A_2(T) \le 2C_0^{\frac12}$, $A_3 \le 2C_0^{\delta_0}$.

$\Rightarrow \|\rho\|_{L^\infty L^\infty} \le 2\bar\rho. \Rightarrow \|\rho\|_{L^\infty L^\infty} \le \frac{7}{4}\bar\rho.$

$\int_0^T \Rightarrow [0, \sigma(T)] + [1(T), T]$ (low, high)

$$D_t \rho + \frac{\rho P}{2\mu + \lambda} = -\partial_t \Big(\int_0^t \frac{\rho F}{2\mu+\lambda} \Big) \Rightarrow$$

$b'(t)$

t

ρ

$\int_0^t \|\rho F\|_{L^\infty}$ $[0,T]$

low $\Leftarrow$ interpolation inequality

high $\Leftarrow$ $\int_{t_1}^{t_2} \|F\|_{L^\infty} \to \int_{\sigma(T)}^T \|F\|_{L^\infty}^{\frac83} + 1$ $(t_2 - t_1)$

$t_1, t_2 \in (\sigma(T), T]$ $\quad 1 \quad \frac83 \sim 0$

b

Zlotnik inequality (Proof) $\to$ 2D Kazhikhov

Theorem (Zlotnik). Let $y(t)$ satisfies $y(0) = y_0$ and

$$y'(t) = g(y)(t) + b'(t), \quad t \in [0,T]$$

where $g \in C(\mathbb{R})$, $g(+\infty) = -\infty$, $y, b \in W^{1,1}(0,T)$. $\exists N_0, N_1 \ge 0$, s.t.

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1), \quad \forall\ 0 \le t_1 \le t_2 \le T.$$

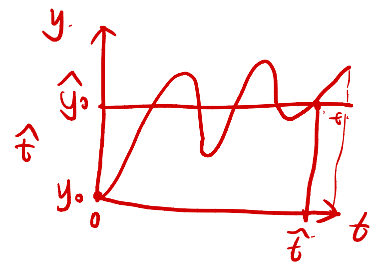
Then $y(t)$ is upper bounded by:

$$y(t) \le \bar y = \max\{y_0, \hat\xi\} + N_0,$$

where $\hat\xi = \hat\xi(N_1)$, defined by $g(\xi) \le -N_1$, $\forall \xi > \hat\xi$.

Proof: $I = \{t\in[0,T] \mid y(t) > \hat{y}_0\}$, $\hat{y}_0 = \max\{y_0, \hat{\xi}\}$

Δ $\quad y(t) \le \hat{y}_0 + N_0$, $\quad \forall t \in I$.



$\forall t\in I$, $y_0 \le \hat{y}_0$, the continuity of y implies $\exists\, \hat{t}\in[0,t)$

$y(\hat{t}) = \hat{y}_0$,

$\forall s\in(\hat{t}, t]$, $y(s) > \hat{y}_0 > \hat{\xi} \Rightarrow g(y(s)) \le -N_1$, $\forall s \in (\hat{t}, t]$.

$y(\hat{t}) = \hat{y}_0$

$$y(t) = y(\hat{t}) = \int_{\hat{t}}^{t} g(y)(s)\,ds + b(t) - b(\hat{t})$$

$$\le \hat{y} - N_1(t_1 - \hat{t}) + N_0 + N_1(t - \hat{t}) \le \hat{y} + N_0 = \bar{y}$$

2D Kazhikhov model: $\|\rho\|_{L^\infty} \leftarrow$ Zlotnik inequality.

$$D_t\theta(\rho) = -p(\rho) + D_t\xi - m. \qquad m = [u^i, R_iR_j](\rho u^j).$$

$$D_t\theta = -p(\theta) + D_t\left(\xi + \int_0^t m\right).$$

$\|\xi\|_{L^\infty}$, $\quad \|m\|_{L^\infty}$.

Theorem. μ is constant, $\lambda = \rho^\beta$, (Kazhikhov model)

$\Rightarrow$ $\beta > \frac{3}{2}$, $\gamma \in (1, 4\beta - 3)$, ✩

(ρ, u) is strong solution $\Rightarrow \|\rho\|_{L^\infty_T L^\infty} \le C$.

$R_T = \|\rho\|_{L^\infty_T L^\infty}$,

$A_1 = \int \mu|\omega|^2 + \frac{F^2}{2\mu+\lambda}$,

$A_2 = \int \rho|\dot{u}|^2$, $\quad A_3 = \int \mu|\omega|^2 + (2\mu + \lambda(\rho))|\nabla\cdot u|^2$.

$$\|\xi\|_{L^\infty} \le C R_T^{\frac{1}{4}} \log^{\frac{1}{2}}(e + A_3^2) + C R_T,$$

$$\|m\|_{L^\infty} \le C_\varepsilon\left(\frac{R_T^{-2(1+k)} A_2^2}{e + A_1^2} + R_T^{\frac{3+k}{2}+\varepsilon} A_3^2 + R_T^{1+\varepsilon}\right), \quad \forall \varepsilon\in(0,1).$$

$$\sup(\log(e + A_1^2 + A_3^2)) + \int_0^T \frac{A_2^2}{e + A_1^2} \le C_\alpha R_T^{1+k+\alpha\beta}, \quad \forall \alpha \in (0,1)$$

$$D_t\theta = -p(\theta) + D_t\left(\xi + \int_0^t m\right)$$

$\Downarrow$ $\qquad$ Zlotnik inequality

$$\|\theta\|_{L^\infty} \le R_T \rightarrow R_T = \|\rho\|_{L^\infty_T L^\infty}$$

$\|\rho\|_{L^\infty}^{\beta} \lesssim \|\theta\|_{L^\infty}$

R_T^{β}

$\theta' = \frac{2\mu + \lambda(\rho)}{\rho}$

$R_T^{\beta} \lesssim R_T^{\bigcirc}$

$\theta \sim 2\mu + \lambda(\rho) \sim \rho^{\beta}$

$\beta \geq \bigcirc$ Δ

β

$R_T^{\beta - \bigcirc} \lesssim 1$

$\Rightarrow R_T \lesssim 1$

$\Rightarrow \|\rho\|_{L_T^\infty L^\infty} \lesssim 1$

1995 Kazhikhov-Vaigant $\beta > 4$ $\beta - \bigcirc > 0$ $\mathbb{T}^2$

2016 Huang-Li $\rightarrow$ $\mathbb{T}^2$ $\beta > \frac{3}{2}$ $(\beta > \frac{4}{3})$ $\mathbb{T}^2$

2016 Li-H. $\rightarrow$ spherical symmetric $\Rightarrow$ $\beta > 1$ $\mathbb{T}^2$ free boundary condition.

Δ

2023 Huang $\rightarrow$ 2D ball. Dirichlet $\rightarrow$ global regularity

Δ

$\|\rho\|_{L^4}$