Pro 2. $u \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$

① $|\hat{u}(x,\tau) - u(x)| \leq \tau^{\alpha} H^{\alpha}_{x}[u : B_\tau(x)]$

② $|D^k \hat{u}(x,\tau)| \leq C(n,\alpha,k,\rho)\,\tau^{\alpha-k} H^{\alpha}_{x}[u, B_\tau(x)]$

$|D\hat{u}(x,\tau)| \leq C\tau^{\alpha-1}\boxed{H^{\alpha}_{x}[u ; B_\tau(x)]} \longrightarrow$

$u \in C^1$. $\quad H^{\alpha}_{x}[u ; B_\tau(x)] = \sup_{\substack{y\in B_\tau(x) \\ y\neq x}} \dfrac{|u(y)-u(x)|}{|y-x|^{\alpha}}$

$= \tau^{1-\alpha}|D\hat{u}(x,\tau)|$.

Pro 3.

(Pro 2) $\boxed{\frac{1}{C}[u]_\alpha \leq \sup_{\tau>0,\, x\in\mathbb{R}^n} \tau^{1-\alpha}|D\hat{u}(x,\tau)|} \leq C[u]_\alpha$

$[u]_\alpha \sim \sup_{\tau>0,\, x\in\mathbb{R}^n} \tau^{1-\alpha}|D\hat{u}(x,\tau)|$

Let $u \in C(\mathbb{R}^n)$. If for $0<\alpha\leq 1$ $\quad R>0$

$\sup_{y\in B_R(x),\, 0<\tau<R} \tau^{1-\alpha}|D\hat{u}(y,\tau)| < \infty \quad (D_x, D_\tau)$.

$\Rightarrow H^{\alpha}_{x}[u ; B_R(x)] < \infty$.

$H^{\alpha}_{x}[u ; B_R(x)] \leq C \sup_{y\in B_R(x),\, 0<\tau<R} \tau^{1-\alpha}|D\hat{u}(y,\tau)|$.

Proof: For $|y-x|<R$, $0<\tau\leq R$.

$|u(x)-u(y)| \leq |\hat{u}(x,\tau)-u(x)| + |\hat{u}(x,\tau)-\hat{u}(y,\tau)| + |\hat{u}(y,\tau)-u(y)|$

① $|\hat{u}(x,\tau)-\hat{u}(y,\tau)| = |D_x\hat{u}(x^*,\tau)|\,|x-y|$.

② $|\tilde{u}(x,\tau) - u(x)|$

We find that when $0<\tau\le R$

$$|\tilde{u}(x,\tau)-u(x)| = \left|\int_{\mathbb{R}^n} \rho(y)u(x-\tau y) - \rho(y)u(x)\,dy\right|.$$

$$= |\tilde{u}(x,\tau) - \tilde{u}(x,0)|$$

$$\tilde{u}(x,\tau) = \int_{\mathbb{R}^n}\rho(y)u(x-\tau y)$$

$$\tilde{u}(x,0) = \int_{\mathbb{R}^n}\rho(y)u(x).$$

$$= \left|\int_0^\tau D_\tau \tilde{u}(x,y)\,dy\right|$$

$$\overset{y=\tau s}{=\!=} \tau\left|\int_0^1 D_\tau \tilde{u}(x,\tau s)\,ds\right|$$

$$= \tau^\alpha \int_0^1 \frac{\underbrace{(\tau y)^{1-\alpha}|D_\tau \tilde{u}(x,y\tau)|}}{y^{1-\alpha}}\,dy.$$

$$\le \sup_{0<\tau<R} \tau^{1-\alpha}|D_\tau \tilde{u}(x,\tau)| \cdot \tau^\alpha\int_0^1 \frac{1}{y^{1-\alpha}}\,dy$$

$$= \frac{1}{\alpha}\tau^\alpha \underbrace{\sup_{0<\tau<R}\tau^{1-\alpha}|D\tilde{u}(x,\tau)|}$$

$\tau=|x-y|$

$$\Rightarrow \underbrace{\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le \left(\frac{2}{\alpha}+1\right)\sup_{\substack{0<\tau<R\\ z\in B_R(x)}} \tau^{1-\alpha}|D\tilde{u}(z,\tau)|}_{\| \atop C}.$$

Conclusion:

① If $u\in C^{0,\alpha}(\mathbb{R}^n) \Rightarrow \exists\, C = C(n,\alpha,\rho)$.

s.t. $\frac{1}{C}[u]_\alpha \le \sup_{\tau>0,\, x\in\mathbb{R}^n} \tau^{1-\alpha}|D\tilde{u}(x,\tau)| \le C[u]_\alpha$.

② $u \in C^{|\beta|+1,\alpha}$

D_x, D_τ.

$\exists\ C(n,\alpha,\beta)$

$$[D^{\beta+1}u]_\alpha \le \sup_{\tau>0}[DD_x^\beta \tilde{u}(x,\tau)]_\alpha^x \le C[D^{\beta+1}u]_\alpha.$$

§3. Potential equation's $C^{2,\alpha}$ estimate

$-\Delta u = f$. $u \in C_0^{2,\alpha}(\mathbb{R}^n)$ $f \in C^{0,\alpha}$

$\Rightarrow\ [D^2u]_\alpha \le C[f]_\alpha$

Lemma 3.1. Let $u \in C^\infty(\mathbb{R}^n)$. satisfy: $-\Delta u = f$.

$-\Delta u$.

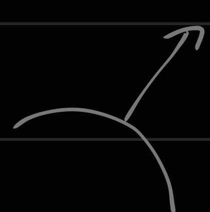
$\Rightarrow\ \forall\ R>0$. $|D_iu(x)| \le \frac{n}{R}\operatorname{osc}_{B_R(x)} u + R\sup_{B_R(x)}|f|$

$(\operatorname{osc}_{B_R(x)} u = \sup_{y\in B_R(x)}|u(y)-u(x)|$.

$|D_iu| \le \frac{n}{R}\operatorname{osc}_{B_R(x)} u$ $-\Delta u = f$.

Proof:

Calculate twice: $\int_{B_\rho} \Delta D_i u\, dx$.



① $\int_{B_\rho} D_i(\Delta u)dx = \int_{\partial B_\rho} \Delta u\, n_i\, ds$ $\cancel{dx_i}$

$\cos(\vec{n}, x_i)ds = dx_1 \overset{\vee}{\cdots} dx_n$

$$= -\int_{\partial B_\rho} f\, n_i\, ds$$

② $$\int_{B_\rho} \Delta(D_i u)\, dx = \int_{B_\rho} \frac{\partial}{\partial \vec{n}} D_i u\, ds$$

$$\underline{\underline{x = \rho y}} \quad \int_{\partial B_1} \frac{\partial D_i u(\rho y)}{\partial \vec{n}} \rho^{n-1}\, ds(y)$$

$$= \underbrace{\rho^{n-1} \frac{\partial}{\partial \rho} \int_{\partial B_1} D_i u(\rho y)\, ds(y)}$$

$$\underbrace{\frac{\partial}{\partial \rho} D_i u(\rho y) = \frac{\partial D_i u(\rho y)}{\partial \vec{n}}}$$

$$= \rho^{n-1} \frac{\partial}{\partial \rho}\left[\rho^{1-n} \int_{\partial B_\rho} D_i u\, ds\right]$$

$$\Rightarrow \rho^{n-1} \frac{\partial}{\partial \rho}\left[\rho^{1-n} \int_{\partial B_\rho} D_i u\, ds\right] = \left| \underbrace{-\int_{\partial B_\rho} f\, n_i\, ds} \right|$$

$0 < \rho \leq R$ $\qquad F_0 = \sup_{B_R(0)} |f|$ $\qquad \leq n w_n \rho^{n-1} F_0$

$$\Rightarrow \pm \int_0^r \frac{\partial}{\partial \rho}\left[\underbrace{\rho^{1-n} \int_{\partial B_\rho} D_i u\, ds}\right] \leq \int_0^r n w_n F_0$$

$$\Rightarrow \pm\left[r^{1-n} \int_{\partial B_r} D_i u\, ds - n w_n D_i u(0)\right] \leq r n w_n F_0$$

$$\xRightarrow{\times\, r^{n-1}}$$

$$\lim_{r \to 0} \frac{1}{r^{n-1}} \int_{\partial B_r} D_i u\, ds = n w_n D_i u(0)$$

$$\pm\left[\underbrace{\int_{\partial B_r}} D_i u\, ds - r^{n-1} n w_n D_i u(0)\right] \leq r^n n w_n F_0$$

$$\pm\left[\int_0^R \int_{\partial B_r} D_i u\, ds - \int_0^R r^{n-1} n w_n D_i u(0)\right] \leq \int_0^R r^n n w_n F_0$$

$$\pm\left[\int_{B_R} D_i u\, dx - R^n w_n D_i u(0)\right] = \frac{n}{n+1} R^{n+1} w_n F_0$$

$B_R(x) \to B_R(0) \quad \Rightarrow |D_i u(0)| \le R F_0 + \frac{1}{\omega_n R^n} \left[\int_{B_R} D_i u \, dx \right]$

$$\int_{B_R} D_i u \, dx = \left| \int_{\partial B_R} (u(x) - u(0)) \, n_i \, dS \right| \le R^{n-1} \operatorname*{osc}_{B_R(0)} u \, n \omega_n$$

$$\Rightarrow |D_i u(0)| \le R F_0 + \frac{n}{R} \operatorname*{osc}_{B_R(0)} u$$

Theorem 3.2.

Let $u \in C_0^{2,\alpha}(\mathbb{R}^n)$ $(0 < \alpha < 1)$ satisfy:

$$-\Delta u = f.$$

$$\Rightarrow [D^2 u]_\alpha \le C(n,\alpha) [f]_\alpha$$

Proof: For fixed $B_R(x_0)$ Let $g(x) = f(x) - f(x_0)$

$$\Rightarrow -\Delta u - f(x_0) = f(x) - f(x_0) = g(x).$$

$$\sup_{B_R(x_0)} |g| \le [f]_\alpha R^\alpha.$$

$-\Delta u - f(x_0) \quad \Rightarrow \quad -\Delta u^\tau(x,\tau) - f(x_0) = g^\tau(x,\tau) \cdot \tau > 0$

$$\int_{\mathbb{R}^n} -\Delta u(y) \rho\left(\frac{x-y}{\tau}\right) \frac{1}{\tau^n} - f(x_0) = g^\tau(x,\tau)$$

$(-\Delta u) \quad -\Delta u^\tau$

$\| u \in C_0^{2,\alpha}(\mathbb{R}^n)$

$$\int_{B_1} -\Delta u(x - \tau z) \rho(z) \, dz.$$

LDC $\Vert$

$$\underbrace{-\Delta \int_{B_1} u(x-\tau z)\,\rho(z)\,dz}$$

$$-\Delta[D^2 u]$$

$$\Rightarrow \quad -\Delta \hat{u}(x,\tau) - f(x_0) = \hat{g}(x,\tau)$$

$$-\Delta\left(D^2 \hat{u}(x,\tau)\right) = D^2 \hat{g}(x,\tau), \quad \underbrace{D^2 = D_x D_\tau}$$

Lemma 3.1

$$\Rightarrow \quad |\underbrace{D_\tau D_x^2 \hat{u}}(x_0,\tau)| \le C\left\{ \frac{1}{R} \underbrace{\operatorname*{osc}_{B_R(x_0)} (D D_x \hat{u}(x,\tau))} + R \sup_{B_R(x_0)} |D^2 \hat{g}| \right\}$$

$[D^2 u]_\alpha$

$$\le C\left\{ \frac{1}{R^{1-\alpha}} [D D_x \hat{u}]_\alpha^x + R \boxed{\sup_{B_R(x_0)} |D^2 \hat{g}|} \right\}$$

Let $\tau \le R$

$$\tau^{1-\alpha} |D_\tau D_x^2 \hat{u}(x_0,\tau)| \le C\left\{ \frac{\tau^{1-\alpha}}{R^{1-\alpha}} \underbrace{[D D_x \hat{u}]_\alpha^x}_{\le C[D^2 u]_\alpha} + \boxed{R\tau^{-1-\alpha} \sup_{B_{R+\tau}(x_0)} |g|} \right\}$$

$\le$

$\frac{1}{C}[D^2 u]_\alpha$

$$\le \frac{R^{1+\alpha}}{\tau^{1+\alpha}} [f]_\alpha$$

$$\| \quad N^{\alpha+1}$$

let $R = N\tau \qquad C N^{\alpha-1} = \frac{1}{2}$

$$\Rightarrow \quad \underbrace{[D^2 u]_\alpha \le C(n,\alpha)\,[f]_\alpha.}$$

Theorem 3.3 Let $u \in C_0^{2,\alpha}(\mathbb{R}^n)$. $\quad -a^{ij} D_{ij} u = f$.

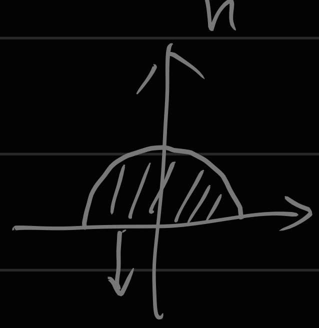
$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2$$

$$\Rightarrow \quad [D^2 u]_\alpha \le \frac{C(n,\alpha,\frac{\Lambda}{\lambda})}{\lambda} [f]_\alpha.$$

Theorem 3.4. The Dirichlet Problem $\mathbb{R}^n_+$,

$u \in C_0^{2,\alpha}(\overline{\mathbb{R}^n_+})$ satisfy: $\begin{cases} -\Delta u = f, & \text{in } \mathbb{R}^n_+ \\ u = 0, & \text{on } \partial\mathbb{R}^n_+ \end{cases}$

$\Rightarrow c(n,\alpha)$ s.t. $[D^2u]_\alpha \le C[f]_\alpha$.



L^2 - estimate. (boundary)

$u \in H_0^1(U)$. $f \in L^2(U)$. $a^{ij} \in C^1(\bar{U})$, $b^i, c \in L^\infty$

$$\underbrace{\int a^{ij} D_i u D_j v + b^i D_i u v + cuv = \int fv}. \quad v \in H_0^1(U).$$

$$v = \underbrace{D_k^{-h}}(\zeta^2 D_k^h u) \qquad k = 1, 2, \dots, n-1$$

$k = n$. $\qquad \underbrace{\|D_{ij} u\|_{L^2}}$.
$i+j < 2n$. $\qquad i, j = n.$ X.

$\underbrace{u \in H^2_{loc}}$ ~

$$-(a^{ij} D_i u)_{x_j} + b^i D_i u + cu = f.$$

$$\underline{D_{nn} u} = \boxed{-\sum_{\substack{i,j \\ i+j<2n}} D_{ij} u + (f - b^i D_i u - cu)} \qquad \underbrace{H^2}_{a.e.}$$

$\| \cdot \|_2$

$$\|D_{nn}u\|_{L^2} \le \|\cdot\|_{L^2}$$

$$\underbrace{[D_{ij}u]_\alpha}_{} \le C[f]_\alpha \qquad [D_{nn}u]$$

$i+j < 2n$

$$D_{nn}u = -\sum_{i=1}^{n-1} D_{ii}u - f$$

$$[D_{nn}u]_\alpha \le C[f]_\alpha$$

Proof: For every $x_0 \in \mathbb{R}^n_+$. $-\Delta u - f(x_0) = g(x)$ $\quad g(x) := f(x) - f(x_0)$

Taking odd extension about $x_n = 0$:

$$v = \begin{cases} u(x', x_n), & x_n \ge 0 \\ -u(x', -x_n), & x_n < 0 \end{cases} \qquad h(x) = \begin{cases} g(x', x_n), & x_n \ge 0 \\ -g(x', -x_n), & x_n < 0 \end{cases}$$

$$f_0(x) = \begin{cases} f(x_0), & x_n > 0 \\ -f(x_0), & x_n < 0 \end{cases}$$

$\Rightarrow v \in W^{2,\infty}(\mathbb{R}^n)$, $\quad \{x_n = 0\}$.

$D_{x'}v \in C^{1,\alpha}(\mathbb{R}^n)$ $\quad x' = (x_1, \dots, x_{n-1})$

$D_{x'x'}v \in C^{0,\alpha}(\mathbb{R}^n)$

$\downarrow$

$$D_{x_n^+} v\Big|_{x_n=0} = \lim_{x_n\to 0^+} \frac{v(x',x_n)-0}{x_n-0} = \underbrace{\lim_{x_n\to 0^+} \frac{u(x',x_n)-0}{x_n-0}}$$

$$D_{x_n^-} v\Big|_{x_n=0} = \lim_{x_n\to 0^-} \frac{v(x',x_n)-0}{x_n-0} = \lim_{x_n\to 0^+} \frac{-u(x',x_n)-0}{-x_n}$$

$$= \underbrace{\lim_{x_n\to 0^+} \frac{u(x',x_n)}{x_n}}.$$

$D_{x'} v \in C^{1,\alpha}(\mathbb{R}^n)$.

$$-\Delta v - f_0(x) = h(x).$$

$$-\Delta \hat{v}(x,\tau) - \hat{f_0}(x,\tau) = \hat{h}(x,\tau).$$

$$\boxed{\hat{v}(x,\tau) = \tau^{-n}\int_{-\infty}^{+\infty} dy_n \int_{\mathbb{R}^{n-1}} \rho_1\left(\frac{x_n-y_n}{\tau}\right)\times \rho_{n-1}\left(\frac{x'-y'}{\tau}\right) v(y)\, dy'}$$

$e^{|x|^2-1}$ $\qquad$ $= \rho\left(\frac{x-y}{\tau}\right)\frac{1}{\tau^n}$

$$-\Delta\left(\underline{DD_{x'}\hat{v}(x,\tau)}\right) = \underline{DD_{x'}\hat{h}(x,\tau)}.$$

$\underline{D_{x'} v \in C^{1,\alpha}(\mathbb{R}^n)}$.

$D_{nn} v \notin C^{0,\alpha}(\mathbb{R}^n)$

$$|D_x DD_{x'}\hat{v}(x,\tau)|$$

$$\le \frac{n}{R}\operatorname*{osc}_{B_R(x_0)} DD_{x'}\hat{v}(x,\tau) + R\sup_{B_R(x_0)} |DD_{x'}\hat{h}(x,\tau)|$$

$$\underline{[DD_{x'}u]_\alpha \le C[f]_\alpha},$$

$$-\Delta u = f \Rightarrow D_{nn}u = -\sum_{i=1}^{n-1} D_{ii}u - f.$$

$$\Rightarrow [D_{nn}u]_\alpha \le C[f]_\alpha.$$

Theorem 3.5.

$u \in C_0^{2,\alpha}(\mathbb{R}^n_+)$ $\quad -a_{ij}D_{ij}u = f$, $\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2$

$u|_{\partial \mathbb{R}^n_+} = 0$

$$\Rightarrow [D^2u]_\alpha \le \frac{C}{\lambda}[f]_\alpha \qquad C = C(n, \alpha, \frac{\Lambda}{\lambda}).$$

$-\Delta u = f$ interior $C_0^{2,\alpha}(\mathbb{R}^n)$ $\quad C_0^{2,\alpha}(\mathbb{R}^n)$.

$\downarrow$

bounded domain interior estimate.

$-\Delta u = f$ in U. $\quad V \subset\subset U$.

$$[D^2u]_{\alpha(V)} \le [f]_{\alpha(U)}$$

$\mathbb{R}^n_+$ (boundary estimate).

§4. Interior Schauder estimate.

In this part, we consider following elliptic equation.

Ω is a bounded domain. $\quad Lu = -a^{ij}D_{ij}u + b^iD_iu + cu = f$ in Ω

$(*)$.

Assume the coefficients satisfy:

$\exists\ \Lambda \geq \lambda > 0$ s.t

$$\lambda |\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n. \quad ①$$

$a^{ij}, b^i, c \in C^{0,\alpha}(\bar{\Omega})$ $(0<\alpha<1)$ and

$$\frac{1}{\lambda}\left\{ \sum_{i,j} \|a^{ij}\|_{\alpha;\Omega} + \sum_i \|b^i\|_{\alpha;\Omega} + \|c\|_{\alpha;\Omega} \right\} \leq \Lambda_\alpha \quad ②$$

Theorem 4.1. (Interior estimate).

Assume ① ② $\vee$ $u \in C^{2,\alpha}(\Omega)$

is the solution of $(*)$. Then for every $\Omega' \subset\subset \Omega$.

we have:

$$\|u\|_{2,\alpha;\Omega'} \leq C(n,\alpha,\frac{\Lambda}{\lambda},\Lambda_\alpha,\Omega',\Omega)\left[\frac{1}{\lambda}\|f\|_{\alpha;\Omega} + \|u\|_{0;\Omega}\right]$$

$$\|u\|_{2;\Omega}, \quad \|u\|_{1;\Omega}.$$

$$\|u\|_{2;\Omega} \leq \varepsilon [u]_{2,\alpha} + C_\varepsilon \|u\|_0.$$

$$\Rightarrow \|u\|_{2,\alpha;\Omega'} \leq C\left[\frac{1}{\lambda}\|f\|_{\alpha;\Omega} + \|u\|_{0;\Omega}\right].$$

$$\leq \|\varphi\|_{\alpha;\Omega}.$$

Lemma 4.2 (Iteration Lemma).

Let $\varphi(t)$ is a bounded nonnegative function on $[T_0,T_1]$

$T_1 > T_0 \geq 0$. For every s.t:

$T_0 \leq t < s \leq T_1$, φ satisfy:

$$\varphi(t) \leq \theta\varphi(s) + \frac{A}{(s-t)^\alpha} + B.$$

$\theta, A, B, \alpha \geq 0 \qquad \theta < 1$

$$\Rightarrow \varphi(\rho) \leq C\left[\frac{A}{(R-\rho)^\alpha} + B\right], \quad \forall\ T_0 \leq \rho < R \leq T_1. \quad C = C(\alpha, \theta)$$

proof,