Pro 2.

Let $u \in C^{\alpha}_{loc}(\mathbb{R}^n)$.

$\Rightarrow$ ① $|\tilde{u}(x,\tau) - u(x)| \le \tau^{\alpha} H^{\alpha}_{x}[u; B_\tau(x)]$.

② $|D^k \tilde{u}(x,\tau)| \le c(n,\alpha,k,\rho)\, \tau^{\alpha-k} H^{\alpha}_{x}[u; B_\tau(x)]$.

Pro 3. Let $u \in C(\mathbb{R}^n)$ If for $0<\alpha\le 1$. $R>0$.

$$\sup_{y\in B_R(x),\, 0<\tau<R} \tau^{1-\alpha} |D\tilde{u}(y,\tau)| < \infty \quad \to D_x, D_\tau.$$

$\Rightarrow H^{\alpha}_{x}[u; B_R(x)] < \infty$.

(u is Hölder continuums with index α at x).

$$H^{\alpha}_{x}[u; B_R(x)] \le C \sup_{\substack{y\in B_R(x)\\ 0<\tau<R}} \tau^{1-\alpha} |D\tilde{u}(y,\tau)|.$$

Proof: For $|x-y|<R$. $0<\tau\le R$.

$$|u(x)-u(y)| \le |\tilde{u}(x,\tau)-u(x)| + |\tilde{u}(x,\tau)-\tilde{u}(y,\tau)| + |\tilde{u}(y,\tau)-u(y)|.$$

Note that: when $0<\tau\le R$.

$$|\tilde{u}(x,\tau) - u(x)| = \left|\int_{\mathbb{R}^n} \rho(y)\, u(x-\tau y) - \rho(y)\, u(x)\, dy\right|$$

$$= |\tilde{u}(x,\tau) - \tilde{u}(x,0)|.$$

$$= \left|\int_0^{\tau} D_\tau \tilde{u}(x,y)\, dy\right| \overset{y=\tau s}{=\!=} \tau \left|\int_0^1 D_\tau \tilde{u}(x,\tau s)\, ds\right|$$

$$\le \tau^{\alpha} \int_0^1 \frac{(\tau y)^{1-\alpha} |D_\tau \tilde{u}(x,y\tau)|}{y^{1-\alpha}}\, dy.$$

$$\le \sup_{0<\tau<R} (y\tau)^{1-\alpha} |D_\tau \tilde{u}(x,y\tau)| \cdot \tau^{\alpha} \int_0^1 \frac{1}{y^{1-\alpha}}\, dy.$$

$$= \frac{\tau^{\alpha}}{\alpha} \sup_{0<\tau<R} \tau^{1-\alpha} |D_\tau \tilde{u}(x,\tau)|.$$

$$\Rightarrow |u(x)-u(y)| \le \frac{2}{\alpha} \tau^{\alpha} \sup_{\substack{0<\tau<R\\ z\in B_R(x)}} \tau^{1-\alpha} |D\tilde{u}(z,\tau)| + |D_x \tilde{u}(x^*,\tau)|\, |x-y|.$$

$\tau = |x-y|$.

$$\Rightarrow y\in B_\tau(x) \text{ 时}. \quad \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \le \left(\frac{2}{\alpha}+1\right) \sup_{\substack{0<\tau<R\\ z\in B_R(x)}} \tau^{1-\alpha} |D\tilde{u}(z,\tau)|.$$

Conclusion:

① If $u \in C^{0,\alpha}(\mathbb{R}^n)$. $\Rightarrow \exists\, C = C(n,\alpha,\rho)$.

s.t. $\frac{1}{C}[u]_\alpha \le \sup_{\tau>0,\, x\in\mathbb{R}^n} \tau^{1-\alpha} \cdot |D\tilde{u}(x,\tau)| \le C[u]_\alpha$.

Proof: $\frac{|D^{\beta+1}_x \tilde{u}(x,\lambda) - D^{\beta+1}_x \tilde{u}(y,\lambda)|}{|x-y|^{\alpha}} \le \sup_{\tau>0} [D^{\beta+1}_x \tilde{u}]^{x}_{\alpha}$.

Let $\lambda\to 0$ $\Rightarrow [D^{\beta+1}_x u]_\alpha \le \sup_{\tau>0} [D^{\beta+1}_x \tilde{u}]^{x}_{\alpha}$.

② $u \in C^{|\beta|+1,\alpha}$.

$\exists\, C(n,\alpha,\rho)$.

令 $y = x+h$. $A_h u = u(x+h)$.

$$[D^{\beta+1}u]_\alpha \le \sup_{\tau>0} [DD^{\beta}_x \tilde{u}(x,\tau)]^{x}_{\alpha} \le C[D^{\beta+1}u]_\alpha.$$

$|DD^{\beta}_x \tilde{u}(x,\tau) - DD^{\beta}_x \tilde{u}(y,\tau)|$.

$= |DD^{\beta}_x (\tilde{u} - \widetilde{A_h u})(x,\tau)|$.

$= |DD^{\beta}_x \widetilde{(u - A_h u)}(x,\tau)| \underset{\alpha=1}{\overset{k=1}{\le}} CH^{1}_{x}[D^{\beta}(u-A_h u)]$

$\le C[D^{\beta+1}(u-A_h u)]_0$.

$\Rightarrow \sup_{\tau>0} [DD^{\beta}_x \tilde{u}(x,\tau)]^{x}_{\alpha} \le C[D^{\beta+1}u]_\alpha$

§3. Potential equation's $C^{2,\alpha}$ estimation.

$-\Delta u = f$. $u \in C^{2,\alpha}_0(\mathbb{R}^n)$. $f \in C^{0,\alpha}$.

$\Rightarrow [D^2 u]_\alpha \le C[f]_\alpha$.

Lemma 3.1. Let $u \in C^{\infty}(\mathbb{R}^n)$ satisfy: $-\Delta u = f$.

$\Rightarrow \forall R>0$. $|D_i u(x)| \le \frac{n}{R} \operatorname*{osc}_{B_R(x)} u + R \sup_{B_R(x)} |f|$ $\qquad \operatorname{osc} u = \sup_{x\in R} |u(x) - u(0)|$

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proof:

Calculate twice:

① $\int_{B_\rho} \Delta D_i u\, dx = \int_{B_\rho} D_i(\Delta u) dx = \int_{\partial B_\rho} (\Delta u) n_i ds = -\int_{\partial B_\rho} f n_i ds.$

② $\int_{B_\rho} \Delta(D_i u) dx = \int_{\partial B_\rho} \frac{\partial}{\partial \vec{n}} D_i u\, ds \xrightarrow{x=\rho y} \int_{\partial B_1} \frac{\partial D_i u(\rho y)}{\partial \vec{n}} \rho^{n-1} ds(y)$

$= \rho^{n-1} \int_{\partial B_1} \frac{\partial D_i u(\rho y)}{\partial \vec{n}} ds.$

$= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{\partial B_1} D_i u(\rho y) ds.$

$= \rho^{n-1} \frac{\partial}{\partial \rho} [\rho^{1-n} \int_{\partial B_\rho} D_i u\, ds]$

$\Rightarrow \rho^{n-1} \frac{\partial}{\partial \rho} [\rho^{1-n} \int_{\partial B_\rho} D_i u\, ds] = -\int_{\partial B_\rho} f n_i ds.$

$F_0 = \sup_{B_R(x_0)} |f|.$

$\Rightarrow \pm \frac{\partial}{\partial \rho} [\rho^{1-n} \int_{\partial B_\rho} D_i u\, ds] \le n w_n F_0.$

$\Rightarrow \pm \int_0^r \frac{\partial}{\partial \rho} [\rho^{1-n} \int_{\partial B_\rho} D_i u\, ds] d\rho \le r n w_n F_0.$

$\Rightarrow \pm [r^{1-n} \int_{\partial B_r} D_i u\, ds - n w_n D_i u(0)] \le r n w_n F_0.$

$\Rightarrow \int_0^R \pm [\int_{\partial B_r} D_i u\, ds - r^{n-1} n w_n D_i u(0)] \le \int_0^R r^n n w_n F_0.$

$\cancel{\Rightarrow |D_i u(0)| \le r F_0 + \frac{1}{w_n r^n}}$

$\Rightarrow |R^n w_n D_i u(0)| \le |\int_{B_R} D_i u\, dx| + \frac{n}{n+1} R^{n+1} F_0 w_n$

$\Rightarrow |D_i u(0)| \le R F_0 + |\int_{\partial B_R} (u(x)-u(0)) n_i ds| \frac{1}{w_n R^n}$

$\le R F_0 + \frac{n}{R} \operatorname{osc}_{B_R} u.$

Theorem 3.2. Let $u \in C_0^{2,\alpha}(\mathbb{R}^n)$, satisfy:

$$-\Delta u = f.$$

$\Rightarrow [D^2 u]_\alpha \le C(n,\alpha) [f]_\alpha.$

proof: For fixed $B_R(x_0)$. let $g(x) = f(x) - f(x_0)$.

$\Rightarrow -\Delta u - f(x_0) = f(x) - f(x_0) = g(x).$

$\Rightarrow \sup_{B_R(x_0)} |g| \le [f]_\alpha R^\alpha.$

$\Rightarrow \int_{\mathbb{R}^n} -\Delta u(y) \rho(\frac{x-y}{\tau}) \frac{1}{\tau^n} - f(x_0) = \check{g}(x,\tau).$

$= \int_{B_1} -\Delta u(x-\tau z) \rho(z) dz \overset{LDC}{=} -\Delta \int_{B_1} u(x-\tau z) \rho(z) dz.$

$\Rightarrow -\Delta \check{u}(x,\tau) - f(x_0) = \check{g}(x,\tau).$

$-\Delta D^2 \check{u}(x_0,\tau) = D^2 \check{g}(x_0,\tau).$ $D^2 = D_x D_\tau.$

$\Rightarrow |D D_x^2 \check{u}(x_0,\tau)| \le C\{\frac{1}{R} \operatorname{osc}_{B_R(x_0)} D D_x \check{u}(x,\tau) + R \sup_{B_R(x_0)} |D^2 \check{g}|$

$\le C\{\frac{1}{R^{1-\alpha}} [D D_x \check{u}]_\alpha^x + R \sup_{B_R(x_0)} |D^2 \check{g}|\}.$

Let $\tau \le R$.

$\Rightarrow \tau^{1-\alpha} |D D_x^2 \check{u}(x_0,\tau)| \le C\{\frac{\tau^{1-\alpha}}{R^{1-\alpha}} [D D_x \check{u}]_\alpha^x + R \tau^{-1-\alpha} \sup_{B_{R+\tau}(x_0)} |g|\}.$

Let $R = N\tau$. $\Rightarrow \tau^{1-\alpha} |D D_x^2 \check{u}(x_0,\tau)| \le C\{N^{\alpha-1} [D^2 u]_\alpha + N^{1+\alpha} [f]_\alpha\}$

Let $C N^{\alpha-1} = \frac{1}{2}.$

$\Rightarrow [D^2 u]_\alpha \le C(n,\alpha) [f]_\alpha,$

Theorem 3.3. $-a^{ij}D_{ij}u = f$

$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2 . \quad \Lambda \ge \lambda > 0 .$$

$$u \in C_0^{2,\alpha}(\mathbb{R}^n) . \quad [D^2u]_\alpha \le \frac{C(n,\alpha,\frac{\Lambda}{\lambda})}{\lambda}[f]_\alpha .$$

Theorem 3.4. The Dirichlet Problem $\mathbb{R}_+^n$.

$u \in C_0^{2,\alpha}(\overline{\mathbb{R}_+^n})$. satisfy: $\begin{cases} -\Delta u = f . & \text{in } \mathbb{R}_+^n . \\ u = 0 . & \text{on } \partial\mathbb{R}_+^n . \end{cases}$

$\Rightarrow \exists\, C(n,\alpha)$ s.t. $[D^2u]_\alpha \le C[f]_\alpha$.

Proof: For every $x_0 \in \mathbb{R}_+^n$. $-\Delta u - f(x_0) = g(x)$. $g(x) = f(x) - f(x_0)$.

Taking odd extension about $x_n = 0$:

$v = \begin{cases} u(x', x_n) , & x_n \ge 0 . \\ -u(x', -x_n) , & x_n < 0 . \end{cases}$ $\quad h(x) = \begin{cases} g(x', x_n) , & x_n \ge 0 . \\ -g(x', -x_n) , & x_n < 0 . \end{cases}$

$f_0(x) = \begin{cases} f(x_0) , & x_n > 0 . \\ -f(x_0) , & x_n < 0 . \end{cases}$

$\Rightarrow v \in W^{2,\infty}(\mathbb{R}^n)$. $\quad D_{x'}v \in C^{1,\alpha}(\mathbb{R}^n)$.

$\Rightarrow -\Delta v - f_0(x) = h(x)$, $\because$ in $\mathbb{R}^n \setminus \{x_n = 0\}$.

$$D_{x_n} v\Big|_{x_n=0} = \lim_{x_n \to 0^+} \frac{u(x', x_n) - 0}{x_n - 0} = \lim_{x_n \to 0^-} \frac{v(x', x_n) - 0}{x_n} .$$

$$= \lim_{x_n \to 0^+} \frac{-u(x', x_n) - 0}{-x_n}$$

$$= \lim_{x_n \to 0^+} \frac{u(x', x_n)}{x_n} .$$

$\Rightarrow -\Delta\tilde{v}(x,\tau) - \tilde{f}_0(x,\tau) = \tilde{h}(x,\tau)$.

$$\tilde{v}(x,\tau) = \tau^{-n}\int_{-\infty}^{+\infty} dy_n \int_{\mathbb{R}^{n-1}} \rho_1\left(\frac{x_n - y_n}{\tau}\right) \rho_{n-1}\left(\frac{x' - y'}{\tau}\right) v(y)\, dy' .$$

$\Rightarrow -\Delta D D_{x'} \tilde{v}(x,\tau) = D D_{x'} \tilde{h}(x,\tau)$.

$$|D_x D D_{x'} \tilde{v}(x_0,\tau)| \le \frac{n}{R} \operatorname*{osc}_{B_R(x_0)} D D_{x'} \tilde{v}(x,\tau) + R \sup_{B_R(x_0)} |D D_{x'} \tilde{h}(x,\tau)|$$

$$\le C\Big[R^{\alpha-1}[D D_{x'} v]_\alpha + R\tau^{-2} \sup_{B_{R+\tau}(x_0)} |h| \Big].$$

$\Rightarrow [D D_{x'} u]_\alpha \le C[f]_\alpha$.

$\because D_{nn} u = -\sum_{i=1}^{n-1} D_{ii} u - f$.

$$[D_{nn} u]_\alpha \le C[f]_\alpha .$$

Theorem 3.5. Assume that $-a_{ij}D_{ij}u = f$. $\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2$.

$u \in C_0^{2,\alpha}(\overline{\mathbb{R}_+^n})$. $\quad u\big|_{\partial\mathbb{R}_+^n} = 0$.

$\Rightarrow [D^2u]_\alpha \le \frac{C}{\lambda}[f]_\alpha \quad C = C(n,\alpha,\frac{\Lambda}{\lambda})$.

§4. ~~Schauder~~ Interior Schauder estimate.

~~Lemma~~ In this part, we consider following elliptic equation:

Ω is a bounded domain. $Lu := -a^{ij}D_{ij}u + b^iD_iu + cu = f$ in Ω. $(*)$

Assume the coefficient satisfy:

$\exists\ \Lambda \geq \lambda > 0$ s.t.

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2,\ \forall x \in \Omega,\ \xi \in \mathbb{R}^n. \quad ①$$

$a^{ij}, b^i, c \in C^{0,\alpha}(\bar{\Omega})$. $(0<\alpha<1)$ and.

$$\frac{1}{\lambda}\Big\{\sum_{i,j}|a^{ij}|_{\alpha;\Omega} + \sum_i |b^i|_{\alpha;\Omega} + |c|_{\alpha;\Omega}\Big\} \leq \Lambda_\alpha. \quad ②$$

Theorem 1. (Interior estimate). Assume ① ② ✓, $u \in C^{2,\alpha}(\Omega)$.

is the solution of $(*)$. Then for every $\Omega' \subset\subset \Omega$, we have.

$$|u|_{2,\alpha;\Omega'} \leq C\big(n,\alpha,\tfrac{\Lambda}{\lambda},\Lambda_\alpha, \mathrm{dist}(\Omega',\partial\Omega)\big)\Big[\frac{1}{\lambda}|f|_{\alpha;\Omega} + |u|_{0;\Omega}\Big]$$

Lemma 4.2. (Iteration lemma). Let $\varphi(t)$ is a bounded nonnegative.

function on $[T_0, T_1]$, where $T_1 > T_0 \geq 0$. For every s, t:

$T_0 \leq t < s \leq T_1$, φ satisfy:

$$\varphi(t) \leq \theta\varphi(s) + \frac{A}{(s-t)^\alpha} + B.$$

$\theta, A, B, \alpha \geq 0$. $\theta < 1$.

$$\Rightarrow \varphi(\rho) \leq C\Big[\frac{A}{(R-\rho)^\alpha} + B\Big]. \quad \forall\, T_0 \leq \rho < R \leq T_1.$$

$C = C(\alpha, \theta)$.