Kazhikov model 2. 2D.

2012 classic, 3D small data. ✓

2016 Kazhikov, 2D large data..

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u) + \nabla p = \mu \Delta u + \nabla(\mu + \underbrace{\lambda(\rho)}_{\lambda=\rho^\beta})(\nabla\cdot u). \end{cases}$$

$p(\rho) = \rho^\gamma$.

Zlotnik inequality: estimate uniform bound of ρ.

$$y'(t) = g(y) + b'(t). \quad \begin{cases} g(y) \to -\infty, \text{ as } y \to \infty. \\ b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1). \end{cases}$$

Kazhikov structure: $F = (2\mu + \lambda(\rho))\nabla\cdot u - p$

$$\nabla\cdot u = -\frac{1}{\rho} D_t \rho.$$

$\downarrow$

$$(2\mu + \lambda(\rho))\nabla\cdot u - p = -\frac{2\mu + \lambda(\rho)}{\rho} D_t \rho - p$$

$\theta'(\rho)$

$\downarrow$

$$F = D_t \theta - p$$

$\downarrow$

$$\Delta^{-1}(\nabla\cdot(\rho \dot u)) = D_t\theta - p$$

$\downarrow$

$$\partial_t(\underbrace{\Delta^{-1}\nabla\cdot(\rho u)}_{\xi}) + \Delta^{-1}\nabla(\nabla\cdot(\rho u \otimes u)) = -D_t\theta - p$$

$\partial_t \xi$

$\downarrow$

$$D_t \xi - u\cdot\nabla\xi + \Delta^{-1}\nabla\cdot(\rho u \otimes u)$$

$\downarrow$

$$[u^i, R_i R_j](\rho u^j) = m.$$

$$\rho D_t u + \nabla p = \nabla((2\mu+\lambda)\nabla\cdot u) - \mu \nabla\times(\underbrace{\nabla\times u}_{\omega})$$

$\downarrow$

$$\rho D_t u = \nabla F - \mu \nabla\times\omega$$

$$\nabla\cdot(\rho D_t u) = \Delta F,$$

$$\nabla\times(\rho D_t u) = \Delta \omega.$$

$$(\underbrace{\Delta F = \nabla\cdot(\rho \dot u)})$$

$\downarrow$

$$F = \Delta^{-1}(\nabla\cdot(\rho\dot u)). \quad (\bar F = 0)$$

$F - \bar F$

$\downarrow$

$\bar F - \bar P$

$\int F(R)$

$$\|F\| + \|\omega\| \lesssim \|\rho \dot u\|_{L^2}$$

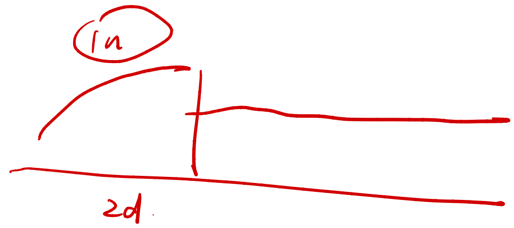
$$D_t\theta = \underbrace{-p}\ \underbrace{- D_t\xi - m(+B)}. = \underbrace{A + b}. \quad \xi - \underbrace{\int_0^t m}_{0} - \underbrace{\int_0^t B}.$$

$$\underbrace{p \sim \rho \sim \theta}$$

$$\theta \to \infty \Rightarrow p \to \infty \Rightarrow -p \to -\infty$$

Kato-Ponce (commutator estimate: $\mathbb{R}^d / \mathbb{T}^d$).

Estimate of $\int_0^t m \Leftarrow \int_0^t \|m\|_{L^\infty} \Leftarrow \underbrace{\|m\|_{L^\infty}}$.



$\dot H^1 \hookrightarrow \underline{BMO}$

2d. $\|\nabla u\|_{L^2} \gtrsim \|u\|_{BMO}$ $\quad \dot H^1 \not\hookrightarrow \underline{L^\infty}$ $\quad$ 2d. $\|u\|_{L^\infty} \lesssim \underline{\|\nabla u\|_{L^2}}$.

$\downarrow \qquad \downarrow \qquad \uparrow$

$\underbrace{1-\frac{2}{2}=0} \qquad 0$

$m = [u^i, R_iR_j](\rho u^j)$. $\swarrow$ Kato-Ponce

$\|m\|_{L^q} \lesssim \underbrace{\|u\|_{BMO}} \|\rho u\|_{L^q} \lesssim \|\nabla u\|_{L^2} \|\rho u\|_{L^q}$.

$\|\nabla m\|_{L^p} \lesssim \|\nabla u\|_{L^4} \|\rho u\|_{L^q}$

$$\|m\|_{L^\infty} \le C(q) \|m\|_{L^q}^{1-\theta} \|\nabla m\|_{L^p}^{\theta}$$
$$\le C(q) \left(\|\nabla u\|_{L^2} \|\rho u\|_{L^q}\right)^{1-\theta} \left(\|\nabla u\|_{L^4} \|\rho u\|_{L^q}\right)^{\theta}$$
$$\le C(q) \|\nabla u\|_{L^2}^{1-\theta} \|\nabla u\|_{L^4}^{\theta} \|\rho u\|_{L^q}$$

$$\begin{cases} 0 = (1-\theta)(-\frac{2}{q}) + \theta(1-\frac{2}{p}), \\ 1-\frac{2}{p} = (1-\frac{2}{4}) + (-\frac{2}{q}). \end{cases} \Rightarrow \begin{cases} p = \frac{4q}{q+4}, \\ \theta = \frac{4}{q}. \end{cases}$$

$$\|m\|_{L^\infty} \lesssim C(q) \|\nabla u\|_{L^2}^{1-\frac{4}{q}} \|\nabla u\|_{L^4}^{\frac{4}{q}} \|\rho u\|_{L^q}.$$

Estimates:

(energy estimate): $\int \rho u^2 \lesssim C \quad \|\rho^{\frac{1}{2}} u\|_{L^2}$.

① $\|\nabla u\|_{L^4} \lesssim \|\nabla\cdot u\|_{L^4} \oplus \|\nabla\times u\|_{L^4} \lesssim \|F\|_{L^4} \oplus \|P\| \oplus \|\omega\| \lesssim A_i \oplus R_T$.

② $\|\rho u\|_{L^q} \lesssim \|\rho^{\frac{1}{2}} u\|_{L^2} \oplus \|u\|_{L^\infty} \lesssim \|\rho^{\frac{1}{2}} u\|_{L^2} \oplus \|\nabla u\|_{L^2} \oplus \log(e + \|\nabla u\|_{L^4}) \lesssim A_i \oplus R_T$.

$\Delta \qquad \Delta$

$R_T = \|\rho\|_{L^\infty_T L^\infty}$,

$A_1^2 = \int \mu|\omega|^2 + \frac{F^2}{2\mu+\lambda}$, $A_2^2 = \int \rho|\dot u|^2$, $A_3^2 = \int \mu|\omega|^2 + (2\mu+\lambda)|\nabla\cdot u|^2$, $\underbrace{\varphi_\alpha = 1 + \|\frac{P - \bar{P}}{2\mu+\lambda}\|_{L^2} + A_1 R_T^{\frac{\alpha\beta}{\gamma}}}$

Estimate $\|\nabla u\|_{L^\varphi}$.

$$\|\nabla u\|_{L^\varphi} \lesssim \|\nabla\cdot u\|_{L^\varphi} + \|\nabla\times u\|_{L^\varphi}$$

$$\lesssim \left\|\frac{F}{2\mu+\lambda}\right\|_{L^\varphi} + \left\|\frac{P}{2\mu+\lambda}\right\|_{L^\varphi} + \|\omega\|_{L^\varphi}$$

$$\left\|\frac{P}{2\mu+\lambda}\right\|_{L^\varphi} \lesssim \left\|\frac{P}{2\mu+\lambda}\right\|_{L^2}^{\frac12} \left\|\frac{P}{2\mu+\lambda}\right\|_{L^\infty}^{\frac12} \le R_T^{\frac{\gamma-2\beta}{4}} R_T^{\frac{\gamma-\beta}{2}} \lesssim R_T^{\frac{3\gamma-4\beta}{4}}$$

$$\left\|\frac{F}{2\mu+\lambda}\right\|_{L^\varphi} \lesssim \left\|\frac{F}{2\mu+\lambda}\right\|_{L^2}^{\frac{1-\alpha}{2}} \|F\|_{L^{2+\frac{2}{\alpha}}}^{\frac{1+\alpha}{2}}$$

$$\left\|\frac{F}{2\mu+\lambda}\right\|_{L^2}^{\frac12} \left\|\frac{F}{2\mu+\lambda}\right\|_{L^\infty}^{\frac12}$$

$$\lesssim A_1^{\frac{1-\alpha}{2}} \|F\|_{L^2}^{\frac{\alpha}{2}} \|F\|_{H^1}^{\frac12}$$

$$\lesssim A_1^{\frac12} R_T^{\frac{\alpha\beta}{4}} \|F\|_{H^1}^{\frac12} \lesssim C_\alpha \varphi_\alpha^{\frac12} \|F\|_{H^1}^{\frac12}$$

$$\lesssim C_\alpha \varphi_\alpha^{\frac12} \left(\|F\|_{H^1}^{\frac12} + \left\|\frac{P}{2\mu+\lambda}\right\|_{L^\infty}^{\frac12} + \|\omega\|_{H^1}^{\frac12}\right).$$

$$\lesssim C_\alpha \varphi_\alpha^{\frac12} \left(\|\rho\dot u\|_{L^2}^{\frac12} + \left\|\frac{P}{2\mu+\lambda}\right\|_{L^\infty}^{\frac12}\right).$$

$|\dot u|$

$$\rho\dot u + \nabla P = \mu\Delta u + \nabla\big((2\mu+\lambda(\rho))\nabla\cdot u\big)$$

$\dot u\, D_t$ $\qquad A_1^2$

$\dot u\ (\partial_t + \nabla\cdot(u\ ))$

$\rho|\dot u|^2$ $\quad A_2$

$$\frac{d}{dt}A_1^2 + A_2^2 \lesssim$$

Hidden structure of the Kazhikhov model: (estimate $\|\rho \dot{u}\|_{L^2}$).

$$\rho D_t u + \nabla P = \mu \Delta u + \nabla((\mu+\lambda(\rho))(\nabla\cdot u))$$

$$\rho D_t u = \nabla((2\mu+\lambda(\rho))\nabla\cdot u - P) - \mu \nabla\times(\nabla\times u).$$

$$\rho D_t u = \nabla F - \mu \nabla\times \omega.$$

test: $\dot{u}$: to get:

$$\int \rho|\dot{u}|^2 = 2\int \nabla F\cdot \dot{u} - 2\mu\int \nabla\times\omega\cdot \dot{u}$$

$$= -2\int F\,\underbrace{(\nabla\cdot\dot{u})} - 2\mu\int \omega\cdot \underbrace{(\nabla\times\dot{u})}.$$

1) $\nabla\cdot\dot{u} = D_t\left(\frac{F+P}{2\mu+\lambda}\right) - 2\nabla u_1\cdot\nabla^\perp u_2 + |\nabla\cdot u|^2$;

2) $\nabla\times\dot{u} = D_t\omega + \omega\nabla\cdot u$;

$$2\int \rho|\dot{u}| = -\partial_t\left(\mu\int|\omega|^2 + \int\frac{F^2}{2\mu+\lambda}\right)$$

$$+ 4\int F\nabla u_1\cdot\nabla^\perp u_2 - 2\int F|\nabla\cdot u|^2 - 2\mu\int|\omega|^2\nabla\cdot u$$

$$- \int\frac{(\beta-1)\lambda-2\mu}{(2\mu+\lambda)^2}F^2\nabla\cdot u + 2\beta\int\frac{\lambda(P-\bar{P})}{(2\mu+\lambda)^2}F\nabla\cdot u$$

$$- 2\gamma\int\frac{P}{2\mu+\lambda}F\nabla\cdot u - 2(\gamma-1)\int\frac{F}{2\mu+\lambda}\int P\nabla\cdot u.$$

Estimate of 1), 2).

$$\nabla\cdot\dot{u} = \nabla\cdot(u_t + u\cdot\nabla u) = (\nabla\cdot u)_t + \nabla(u\cdot\nabla u)$$

$$= (\nabla\cdot u)_t + u\cdot\nabla(\nabla\cdot u) + (\nabla u)^\perp : \nabla u$$

$$= D_t(\nabla\cdot u) - 2\nabla u_1\cdot\nabla^\perp u_2 + |\nabla\cdot u|^2$$

$$\nabla\times\dot{u} = \nabla\times(u_t + u\cdot\nabla u) = \partial_t(\nabla\times u) + \underbrace{\overset{u\cdot\nabla(\nabla\times u)}{\nabla\times(u\cdot\nabla u)}}.\quad \left(= \nabla\times\left(\tfrac{1}{2}\nabla|u|^2 + (\nabla\times u)\times u\right)\right)$$

$$= \omega_t + \underbrace{\nabla\times(\omega\times u)} = \omega_t + \underbrace{u\cdot\nabla\omega} - \omega\cdot\nabla u\overset{?}{+}\omega\nabla\cdot u - \cancel{u\nabla\cdot\omega}$$

$$= D_t\omega + \omega\nabla\cdot u$$

De-Giorgi $L^p \to L^\infty$

hyperbolic-parabolic $\leftarrow$ Gronwall

Zlotnik

PDE $\to$ ODE