Kazhikhov model's global regularity 3.

$\lambda = \rho^{\beta}$, $(\beta > 0)$.

$$\underline{D_t \theta = -P - D_t \xi - m + B}, \qquad \begin{cases} \theta'(\rho) = \dfrac{2\mu + \lambda(\rho)}{\rho}, \\ P = \rho^{\gamma}, \quad \xi = \Delta^{-1}(\nabla \cdot (\rho u)), \\ m = [u^i, R_i R_j](\rho u^j), \quad B = \bar{P} - \bar{F}, \end{cases}$$

$\sim \rho u^2$

$\|\rho\|_{L^\infty_T L^\infty} = R_T \Rightarrow R_T \le C/C(T)$.

$$\theta'(\rho) = \frac{2\mu + \lambda(\rho)}{\rho} \sim \rho^{\beta - 1}, \quad \Rightarrow \theta(\rho) \sim \rho^{\beta},$$

$$\Rightarrow \quad \rho^{\beta} \lesssim \theta \lesssim R_T^{\alpha} \quad \Rightarrow \quad R_T^{\beta} \lesssim \overset{C(T)}{R_T^{\alpha}}, \quad \text{if } \alpha < \beta,$$

$$\Rightarrow \underline{R_T \lesssim C(T)}.$$

$\int_0^t \|m\|_{L^\infty} \Leftarrow$ most important part:

$$\|m\|_{L^\infty} \le C(q) \|\nabla u\|_{L^2}^{1-\theta} \|\nabla u\|_{L^4}^{1-\theta} \|\rho u\|_{L^q}, \quad (q > 4).$$

① $\|\nabla u\|_{L^4} \lesssim \|F\|_{L^4} \oplus \|P\| \oplus \|\omega\| \lesssim A_i \oplus R_T \lesssim R_T$

② $\|\rho u\|_{L^q} \lesssim \|\rho^{\frac{1}{2}} u\|_{L^2} \oplus \|u\|_{L^\infty} \lesssim \|\rho^{\frac{1}{2}} u\|_{L^2} \oplus \|\nabla u\|_{L^2} \oplus \log(e + \|\nabla u\|_{L^4}) \lesssim A_i \oplus R_T \lesssim R_T$.

$$\underline{\|\omega\|_{L^2}^{\frac{1}{2}} \|\nabla \omega\|_{L^2}^{\frac{1}{2}}} \Leftarrow C_\alpha \varphi_\alpha^{\frac{1}{2}} \|\omega\|_{H^1}^{\frac{1}{2}}$$

$$\|\nabla u\|_{L^4} \le \|\nabla \cdot u\|_{L^4} + \|\nabla \times u\|_{L^4} \lesssim \left\|\frac{F}{2\mu + \lambda}\right\|_{L^4} + \left\|\frac{P}{2\mu + \lambda}\right\|_{L^4} + \underline{\|\omega\|_{L^4}}.$$

$\downarrow \|F\|_{L^\infty}$ ⓧ

$$\left\|\frac{P}{2\mu+\lambda}\right\|_{L^2 \oplus L^\infty} \lesssim R_T^{\frac{3\gamma - 4\beta}{4}}$$

$$\underline{\|F\|_{L^q} \lesssim \|F\|_{BMO} \lesssim \|\nabla F\|_{L^2}}.$$

$$\lesssim \left\|\frac{F}{2\mu + \lambda}\right\|_{L^2}^{\frac{1-\alpha}{2}} \|F\|_{L^{2+\frac{2}{\alpha}}}^{\frac{1+\alpha}{2}} \lesssim C_\alpha \varphi_\alpha^{\frac{1}{2}} \|F\|_{H^1}^{\frac{1}{2}}$$

$$\lesssim C_\alpha \varphi_\alpha^{\frac{1}{2}} \left(\|F\|_{H^1}^{\frac{1}{2}} + \|\omega\|_{H^1}^{\frac{1}{2}} \right).$$

$$\frac{d}{dt} A_1^2 + A_2^2 \lesssim C_\alpha R_T \left(\varphi_\alpha^2 + \|\rho\|_{L^\beta}^\beta R_T^{-2} \right) A_3^2 .$$

$$\boxed{\begin{aligned} &A_1 = \int \mu|\omega|^2 + \int \frac{F^2}{2\mu+\lambda}, \quad A_2 = \int \rho|\dot u|^2, \\ &A_3 = \int \mu|\omega|^2 + \int (2\mu+\lambda)|\nabla\cdot u|^2 . \\ &\varphi_\alpha = 1 + \left\|\frac{P}{2\mu+\lambda}\right\|_{L^3} + A_1 R_T^{\frac{\alpha\beta}{2}} . \end{aligned}}$$

$$\rho D_t u = \nabla F - \mu \nabla \times \omega$$

$\Big\downarrow$ test $\dot u$.

$$2\int \rho|\dot u|^2 = \underline{-2\int F(\nabla\cdot\dot u)} - \underline{2\mu\int \omega\cdot(\nabla\times\dot u)}$$

$$\boxed{\begin{aligned} &\nabla\cdot\dot u = D_t(\nabla\cdot u) - 2\nabla u_1\cdot\nabla^\perp u_2 + |\nabla\cdot u|^2 \\ &\nabla\times\dot u = D_t\omega + \omega\nabla\cdot u . \end{aligned}}$$

Kazhikhov

$\downarrow$ first hidden structure.

☆ $$= -\frac{d}{dt}\left(\mu\int|\omega|^2 + \int\frac{F^2}{2\mu+\lambda} \right) \leftarrow \text{time-term}\quad A_1 .$$

$$+ \underbrace{4\int F\nabla u_1\cdot\nabla^\perp u_2}_{I_2} - \underbrace{2\int F|\nabla\cdot u|^2}_{I_3} - \underbrace{2\mu\int|\omega|^2\nabla\cdot u}_{I_1}$$

$$- \underbrace{\int\frac{(\beta-1)\lambda-2\mu}{(2\mu+\lambda)^2}F^2\nabla\cdot u}_{I_4} + \underbrace{2\beta\int\frac{\lambda(P-\bar P)}{(2\mu+\lambda)^2}F\nabla\cdot u}_{I_5}$$

$$- \underbrace{2\gamma\int\frac{P}{2\mu+\lambda}F\nabla\cdot u}_{I_6} - \underbrace{(2\gamma-1)\int\frac{F}{2\mu+\lambda}\int P\nabla\cdot u}_{B} \leftarrow \text{boundary term} .$$

$I_1 \lesssim C\varphi_\alpha \|F\|_{H^1} A_3$,

☆ $I_2 \lesssim C\varphi_\alpha \|\omega\|_{H^1} A_3$,

$I_3 + \cdots + I_6 \lesssim C_\alpha \varphi_\alpha \|F\|_{H^1} A_3$.

$B \lesssim CA_3\|F\|_{L^2} + CA_3^2$

$$\Longrightarrow \quad \frac{d}{dt}A_1^2 + A_2^2$$

$$\lesssim C_\alpha\varphi_\alpha\left(\|F\|_{H^1} + \|\omega\|_{H^1}\right)A_3 + CA_3^2 .$$

$$\lesssim$$

$$2\int \rho|\dot u|^2 = -2\int F(\nabla\cdot\dot u) - 2\mu\int \omega\cdot(\nabla\times\dot u)$$

$$\boxed{\begin{aligned}\nabla\cdot\dot u &= D_t(\nabla\cdot u) - 2\nabla u_1\cdot\nabla^\perp u_2 + |\nabla\cdot u|^2\\ \nabla\times\dot u &= D_t\omega + \omega\nabla\cdot u\end{aligned}}$$

$$= -2\int F D_t(\nabla\cdot u) - 4\int F\nabla u_1\cdot\nabla^\perp u_2 - 2\int F|\nabla\cdot u|^2$$

① D_t 提出来

$$-2\mu\int\omega\cdot D_t\omega - 2\mu\int\omega\cdot(\omega\nabla\cdot u)$$

$$= -\frac{d}{dt}\left(\int\frac{F^2}{2\mu+\lambda}\right) + \triangle \to \quad = -\frac{d}{dt}\Big(\underbrace{\int\frac{F^2}{2\mu+\lambda} + \mu\int|\omega|^2}_{}\Big)^{A_1} + \triangle$$

$$-\frac{d}{dt}\left(\int\mu\int|\omega|^2\right) + \triangle$$

$$-2\mu\int|\omega|^2\nabla\cdot u$$

$$\int F D_t(\nabla\cdot u) = \int F D_t\left(\frac{F}{2\mu+\lambda}\right) + \int F D_t\left(\frac{P-\bar P}{2\mu+\lambda}\right)$$

$$2\int F D_t\left(\frac{P-\bar P}{2\mu+\lambda}\right) = 2\int F D_t\left(\frac{P}{2\mu+\lambda}\right) - 2\int F\bar P D_t\left(\frac{1}{2\mu+\lambda}\right) - \int\frac{F}{2\mu+\lambda}D_t\bar P$$

$$= 2\int F\left(\frac{\gamma P(2\mu+\lambda) - \beta P\lambda}{(2\mu+\lambda)^2}\right) + 2\beta\int\frac{\lambda\bar P}{(2\mu+\lambda)^2}F\nabla\cdot u + 2(\gamma-1)\,O$$

$$= -2\beta\int\frac{\lambda(P-\bar P)}{(2\mu+\lambda)^2}F\nabla\cdot u + 2\gamma\int\frac{P}{2\mu+\lambda}F\nabla\cdot u + 2(\gamma-1)\int\frac{F}{2\mu+\lambda}\int P\nabla\cdot u$$

$$D_t\bar P = D_t\int P = \int\partial_t P = \int(\rho P' - P)\nabla\cdot u$$

$$2\int F D_t\left(\frac{F}{2\mu+\lambda}\right) = \frac{d}{dt}\int\frac{F^2}{2\mu+\lambda} + \frac{(\beta-1)\lambda-2\mu}{(2\mu+\lambda)^2}F^2\nabla\cdot u$$

$$= 2\int F\partial_t\left(\frac{F}{2\mu+\lambda}\right) + u\cdot\nabla\left(\frac{F}{2\mu+\lambda}\right)F$$

$$= \int\partial_t\left(\frac{F^2}{2\mu+\lambda}\right) + u\cdot\nabla\left(\frac{F^2}{2\mu+\lambda}\right) + F^2D_t\left(\frac{1}{2\mu+\lambda}\right)$$

$$= \int\partial_t\left(\frac{F^2}{2\mu+\lambda}\right) - \left(\frac{F^2}{2\mu+\lambda}\right)\nabla\cdot u + \left(\frac{\lambda\beta}{(2\mu+\lambda)^2}\right)F^2\nabla\cdot u$$

$$= \frac{d}{dt}\int\frac{F^2}{2\mu+\lambda} + \frac{(\beta-1)\lambda-2\mu}{(2\mu+\lambda)^2}F^2\nabla\cdot u$$

$2\int \rho |\dot{u}|^2$

$$= -\frac{d}{dt}\left(\mu\int|w|^2 + \int\frac{F^2}{2\mu+\lambda}\right) \leftarrow \text{time-term } A_1.$$

$$+ \underbrace{4\int F\nabla u_1\cdot\nabla^\perp u_2}_{I_2} - \underbrace{2\int F|\nabla\cdot u|^2}_{I_3} - \underbrace{2\mu\int|w|^2\nabla\cdot u}_{I_1}$$

$$- \underbrace{\int\frac{(\beta-1)\lambda-2\mu}{(2\mu+\lambda)^2}F^2\nabla\cdot u}_{I_4} + \underbrace{2\beta\int\frac{\lambda(P-\bar{P})}{(2\mu+\lambda)^2}F\nabla\cdot u}_{I_5}$$

$$- \underbrace{2r\int\frac{P}{2\mu+\lambda}F\nabla\cdot u}_{I_6} - \underbrace{(2r-1)\int\frac{F}{2\mu+\lambda}\int P\nabla\cdot u}_{B} \leftarrow \text{boundary term.}$$

$$I_1 \lesssim C\varphi_\alpha\|w\|_{H^1}A_3.$$

$$I_2 \lesssim C\varphi_\alpha\|F\|_{H^1}A_3.$$

$$I_3+\cdots+I_6 \lesssim C_\alpha\varphi_\alpha\|F\|_{H^1}A_3.$$

$$B \lesssim CA_3\|F\|_{L^2} + CA_3^2.$$

$$I_1 \leq C\|w\|_{L^4}^2\|\nabla\cdot u\|_{L^2} \leq C\underbrace{\|w\|_{L^2}}\|\nabla w\|_{L^2}\|\nabla u\|_{L^2}.$$

$$\leq \underbrace{A_1}\|w\|_{H^1}A_3 \leq C_\alpha\varphi_\alpha\|w\|_{H^1}A_3,$$

$$I_2 \leq \int|F||\nabla u_1\cdot\nabla^\perp u_2| \leq \|F\|_{BMO}\|\nabla u_1\cdot\nabla^\perp u_2\|_{\mathcal{H}^1} \leq \|\nabla F\|_{L^2}\|\nabla u\|_{L^2}^2$$

$$\leq \underbrace{C\left(\|w\|_{L^2} + \left\|\frac{F}{2\mu+\lambda}\right\|_{L^2} + \left\|\frac{P-\bar{P}}{2\mu+\lambda}\right\|_{L^2}\right)}\|F\|_{H^1}\left(\|w\|_{L^2}+\|\nabla\cdot u\|_{L^2}\right).$$

$$\leq \underbrace{c\left(A_1 + \left\|\frac{P}{2\mu+\lambda}\right\|_{L^2}+1\right)}_{C_\alpha\varphi_\alpha}\ \circ\ \circ \leq C_\alpha\varphi_\alpha\|F\|_{H^1}A_3$$

$$\frac{1}{2+\frac{\beta}{r}} + \frac{1}{2+\frac{4r}{\beta}} = \frac{1}{2}.$$

$$I_5+I_6 = -2r\int\frac{P}{2\mu+\lambda}F\nabla\cdot u \leq C\left\|\frac{P}{(2\mu+\lambda)^{\frac{3}{4}}}\right\|_{L^{2+\frac{\beta}{r}}}\|F\|_{L^{2+\frac{4r}{\beta}}}\|(2\mu+\lambda)^{\frac{1}{4}}\nabla\cdot u\|_{L^2}$$

$$\leq \underline{C_\alpha\varphi_\alpha\|F\|_{H^1}A_3}$$

$$I_4 \leq C\int\frac{F^2}{2\mu+\lambda}|\nabla\cdot u| \leq C\left\|\frac{F^2}{2\mu+\lambda}\right\|_{L^2}\|\nabla u\|_{L^2} \leq C_\alpha\varphi_\alpha\|F\|_{H^1}A_3.$$

$$I_3 \leq C_\alpha\varphi_\alpha\|F\|_{H^1}A_3$$

$$B \le C A_3 \|F\|_{L^2} + C A_3^2$$

$$\frac{d}{dt} A_1^2 + \cancel{2} A_2^2 \le \sum I_i + B \le C_\alpha \varphi_\alpha (\|F\|_{H^1} + \|w\|_{H^1}) A_3 + C A_3^2$$

$$\le C_\alpha \varphi_\alpha (\underbrace{\|\nabla F\|_{L^2} + \|\nabla w\|_{L^2}}_{\lesssim \|\rho \dot u\|_{L^2} \lesssim R_T^{\frac{1}{2}} A_2} + \underbrace{\|F - \bar F\|_{L^2} + |\bar F|}_{\|\rho\|_{L^{2\beta}}^{\frac{\beta}{3}} A_3}) A_3 + C A_3^2$$

$|\bar F| = \int (2\mu+\lambda)(\nabla \cdot u) + \int |P|$ ρ^β

$\le \int (2\mu+\lambda)^{\frac12} (2\mu+\lambda)^{\frac12} (\nabla\cdot u)$ L^2 L^2 $+ \int P$

$\le \|\rho\|_{L^\beta}^{\frac{\beta}{2}} A_3 + P$.

$$\le C_\alpha \varphi_\alpha (R_T^{\frac12} A_2 + \|\rho\|_{L^\beta}^{\frac{\beta}{3}} A_3) A_3 + C A_3^2$$

$$\le A_2 \cdot C_\alpha R_T^{\frac12} \varphi_\alpha A_3 + C_\alpha R_T^{\frac12} \varphi_\alpha A_3 \cdot R_T^{-\frac12} \|\rho\|_{L^\beta}^{\frac{\beta}{2}} A_3 + C A_3^2$$

$$\le A_2^2 + C_\alpha (R_T \varphi_\alpha^2 A_3^2 + R_T^{-1} \|\rho\|_{L^\beta}^{\beta}) A_3^2$$

$$\int \frac{d}{dt} A_1^2 + \int A_2^2 \le C_\alpha R_T (\varphi_\alpha^2 + R_T^{-1} \|\rho\|_{L^\beta}^{\beta}) A_3^2$$

$$\varphi_\alpha \lesssim 1 + R_T^{\frac{\gamma-2\beta}{2}} + R_T^{\frac{\alpha\beta}{2}} A_1$$

$$\le C_\alpha R_T (1 + R_T^{\gamma-2\beta} + R_T^{\alpha\beta} A_1^2 + R_T^{\beta-\gamma-2}) A_3^2 \qquad \int A_3^2 \lesssim 1 .$$

divide $e + A_1^2$,

$$A_1^2 + A_2^2 \le R_T .$$

$$\frac{\frac{d}{dt}(e + A_1^2)}{e + A_1^2} + \frac{A_2^2}{e + A_1^2} \le C_\alpha R_T^{1 + \max\{0, \gamma-2\beta, \beta-\gamma-2\}} A_3^2$$

$$+ C_\alpha R_T^{1+\alpha\beta} A_3^2$$

$$\le C_\alpha R_T^{1 + \max\{0, \gamma - 2\beta, \beta-\gamma-2\} + \alpha\beta} A_3^2$$

$\|\rho\|_{L^\infty} \le C \|e\|_{W^{1,q}} \|\le C$

$\|\rho\|_{L^\infty}$ ∇u u $-$ k bootstrap

$$\lesssim C_\alpha R_T^{1+k+\alpha\beta} A_3^2 \qquad C^\alpha \to C^2$$

$$\Rightarrow \log(e + A_1^2) + \int \frac{A_3^2}{e + A_1^2} \lesssim C_\alpha R_T^{1+k+\alpha\beta}$$

C^α Strong Remark C^{α} $C^{2+\alpha}$