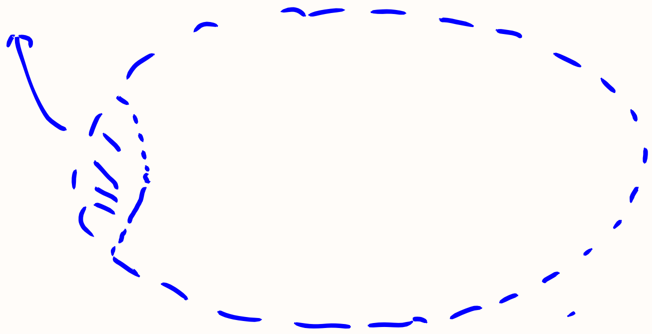
§4. Schauder interior estimate.



$$Lu = -a^{ij}D_{ij}u + b^i D_i u + cu = f \quad \text{in } \Omega.$$

$\Omega' \subset\subset \Omega$.

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \qquad \text{interior estimate.}$$

Assume Ω is a bounded domain in $\mathbb{R}^n$ Consider:

$$Lu = -a^{ij}D_{ij}u + b^i D_i u + cu = f \quad \text{in } \Omega. \quad (4.3)$$

Coefficients satisfy:

$\exists\ \Lambda \geq \lambda > 0$ s.t.

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n \quad (4.4)$$

$a^{ij}, b^i, c \in C^{0,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ and:

$$\frac{1}{\lambda}\Big\{ \sum_{i,j} |a^{ij}|_{\alpha;\Omega} + \sum_i |b^i|_{\alpha;\Omega} + |c|_{\alpha;\Omega} \Big\} \leq \Lambda_\alpha, \quad (4.5)$$

$$|a^{ij}|_{\alpha;\Omega} = |a^{ij}|_0 + [a^{ij}]_\alpha$$

Lemma 4.1.

Let $\varphi(t)$ is a bounded nonnegative function defined on $[T_0, T_1]$,

For every : $T_0 \le t < s \le T_1$, φ satisfy :

$\varphi(s) \le \varphi(s')$

$$\boxed{\varphi(t) \le \boxed{\theta\varphi(s)} + \frac{A}{(s-t)^\alpha} + B}$$

$\varphi(t) \le \cancel{\theta\varphi(s)}$

$\le \cancel{\theta^2\varphi(s_1)}$.

where $\theta < 1$, $\theta . A . B , \alpha \ge 0$ (constant).

$$\Rightarrow \quad \varphi(\rho) \le C\left[\frac{A}{(R-\rho)^\alpha} + B\right], \quad \forall\, T_0 \le \rho < R \le T_1 .$$

$C = C(\alpha, \theta)$

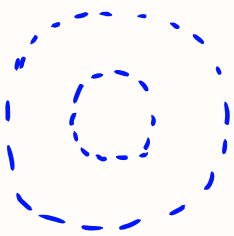
$\varphi(s) = [D^2 u]_{\alpha; B_s}$

$\|u\|_{L^{p_1}(B_r)} \le C \|u\|_{L^{p_2}(B_r)}$.

$p_2 > p_1$

$\|u\|_{L^{p_2}(B_r)} \le C \|u\|_{L^{p_1}(B_R)}$

$r < R$.



Proof:

$t_0 = \rho$, $t_{i+1} = t_i + $ ~~~ $\quad i \to +\infty \quad \lim_{i\to+\infty} t_i = R$

$$t_{i+1} = t_i + \boxed{\frac{1}{2^{i+1}}}(R-\rho) \qquad \rho \to R .$$

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \qquad T = \frac{1}{2} \to \tau .$$

$0<\tau<1$. $\quad t_{i+1} = t_i + (1-\boxed{\tau})\tau^i (R-\rho)$.

$$\sum_{i=0}^{+\infty} \tau^i = \frac{1}{1-\tau}$$

$i \to +\infty \quad t_i \to R$

$$t_{i+1} = t_i + (1-\tau)\tau^i (R-\rho). \qquad t_{i+1} - t_i = (1-\tau)\tau^i (R-\rho)$$

$$\varphi(t_i) \le \theta\,\varphi(t_{i+1}) + \frac{A}{(1-\tau)^\alpha \tau^{i\alpha} (R-\rho)^\alpha} + B.$$

$$\varphi(t_0) \le \theta\,\varphi(t_1) + \frac{A}{(1-\tau)^\alpha (R-\rho)^\alpha} (\tau^\alpha)^{-0} + B.$$

$$\le \theta\left[\theta\,\varphi(t_2) + \frac{A}{(1-\tau)^\alpha (R-\rho)^\alpha} (\tau^\alpha)^{-1} + B\right] + \frac{A}{(1-\tau)^\alpha (R-\rho)^\alpha} (\tau^\alpha)^{-0} + B.$$

$$= \theta^2 \varphi(t_2) + \frac{A}{(1-\tau)^\alpha (R-\rho)^\alpha}\left[\frac{\theta}{\tau^\alpha} + 1\right] + B(1+\theta).$$

$$\le \theta^k \varphi(t_k) + \frac{A}{(1-\tau)^\alpha (R-\rho)^\alpha}\left[\frac{1-\left(\frac{\theta}{\tau^\alpha}\right)^k}{1-\frac{\theta}{\tau^\alpha}}\right] + B\,\frac{1-\theta^k}{1-\theta}.$$

$k \to +\infty$. Let τ s.t. $\theta\tau^{-\alpha} < 1$. $\quad \tau = \tau(\theta)$

$$\to \quad \varphi(\rho) \le C(\alpha,\theta)\left[\frac{A}{(R-\rho)^\alpha} + B\right]. \quad \forall\ T_0 \le \rho < R \le T_1.$$

Lemma 4.2. Let (4.3)'s coefficients satisfy (4.4) and (4.5).

Then there exists $R_0 \le 1$ $\left(R_0 = R_0(n, \alpha, \frac{\Lambda}{\lambda}, \Lambda_\alpha)\right)$. for every $0 < R \le R_0$. If $B_R \subset \Omega$, $u \in C_0^{2,\alpha}(B_R)$ satisfy (4.3).

$$\boxed{\begin{array}{l} u \in C^{2,\alpha}(B_R) \Rightarrow \underline{u \in C^{2,\alpha}(\bar{B}_R)} \\ \forall\, \Omega \subset\subset B_R. \quad \|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq +\infty \end{array}}$$

$$\Rightarrow [D^2u]_{\alpha;B_R} \leq C\left\{\frac{1}{\lambda}\underline{[f]_{\alpha;B_R}} + \frac{1}{R^{2+\alpha}}|u|_{0;B_R}\right\}.$$

$$[f]_{\alpha;\Omega} = \sup_{x_0\in\Omega} H^{\alpha}_{x_0}[u;\Omega].$$

$$H^{\alpha}_{x_0}[u;\Omega] = \sup_{x\in\Omega}\frac{|u(x)-u(x_0)|}{|x-x_0|^{\alpha}} < \infty.$$

$$[f]_{\alpha;\Omega} = \sup_{x,y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < \infty$$

Proof: Without lossing of generality, $\lambda = 1$.

Consider $B_R(x_0)$. $\quad \underline{-\Delta u = f} \quad \underline{u \in C_0^{2,\alpha}(B_R(x_0))}$

$$-a^{ij}(x_0)D_{ij}u = \bar{f}$$

$$\bar{f} = f + (a^{ij}(x) - a^{ij}(x_0))D_{ij}u - b^iD_iu - cu.$$

$$[D^2u]_\alpha \leq C\underline{[\bar{f}]_\alpha}$$

$$[uv]_\alpha \leq |u|_0[v]_\alpha + [u]_\alpha|v|_0 \leq |u|_\alpha\cdot|v|_\alpha \quad ☆$$

$$[\tilde f]_\alpha \le \underset{①}{[f]_\alpha} + \underset{②}{[(a^{ij}(x)-a^{ij}(x_0))D_{ij}u]_\alpha} + \underset{③}{[b^i D_i u]_\alpha} + \underset{④}{[cu]_\alpha}$$

$[0,2]$ $\uparrow$ $(2, 2+\alpha)$

$(0, 2+\alpha)$

② $\le [D_{ij}u]_\alpha \, |a^{ij}(x)-a^{ij}(x_0)|_0 + |D^2u|_0 \, [a^{ij}(x)-a^{ij}(x_0)]_\alpha$

$|a^{ij}|_\alpha \le \Lambda_\alpha$.

$u \in C^1$

$\le \Lambda_\alpha [D^{ij}u]_\alpha + \Lambda_\alpha |D^2u|_0$.

$|Du|$

$|D^2u|$

$[u]_\alpha \le |u|_0 + |Du|_0$

$\to 2+\alpha$

③ $\le [b^i]_\alpha |D_iu| + |b^i|_0 \boxed{[D_iu]_\alpha}$.

$[D_iu]_\alpha$

$|Du| \le \varepsilon [D^2u]_\alpha + C(\varepsilon)|u|_0$

$|D^2u| \le \varepsilon [D^2u]_\alpha + C(\varepsilon)|u|_0$

$[D^2u]_\beta \le \varepsilon [D^2u]_\alpha + C(\varepsilon)|u|_0$.

$\le \Lambda_\alpha |Du|_0 + \Lambda_\alpha [\varepsilon [D^2u]_\alpha + C(\varepsilon)|u|_0]$.

④ $\le [c]_\alpha |u|_0 + [u]_\alpha |c|_0$

$\le C|u|_0 + C[\varepsilon [D^2u]_\alpha + c(\varepsilon)|u|_0]$

$\varepsilon = \frac{R}{2}$.

$[u]_2 \le R^\alpha [u]_{2,\alpha} + \frac{C}{R^2}|u|_0$ $2+\alpha$.

$2+\alpha-1$

$[u]_1 \le R^{1+\alpha}[u]_{2,\alpha} + \frac{C}{R}|u|_0$.

$$\boxed{[u]_\alpha \le R^2 [u]_{2,\alpha} + \frac{C}{R^\alpha}|u|_0}$$

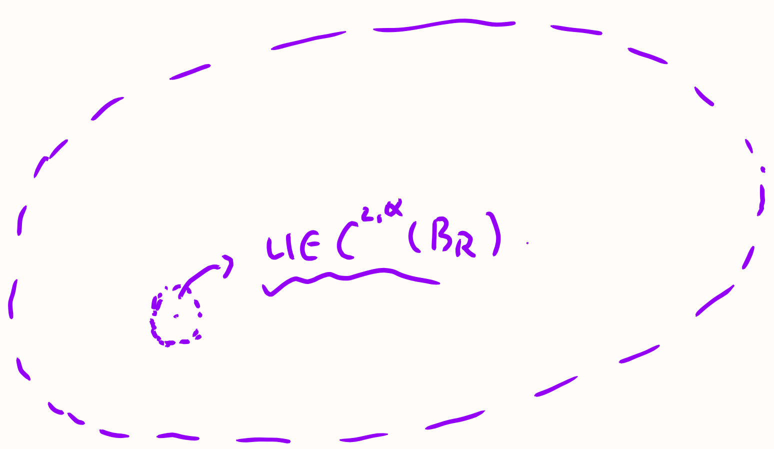
$2+\alpha-\beta$. β.

$\Rightarrow$

$[D^2u]_\alpha \le CR^\alpha[D^2u]_\alpha +$
$C\{[f]_\alpha + \frac{1}{R^2}|u|_0\}$.

$R_0 = \left(\frac{1}{2C}\right)^{\frac{1}{\alpha}}$.

$\Rightarrow$ when $0 < R \le R_0$. $[D^2u]_\alpha \le C\{[f]_\alpha + \frac{1}{R^2}|u|_0\}$



$u \in C^{2,\alpha}(B_R)$.

$v = \beta u$.

$v \in C_0^{2,\alpha}$.

$\theta < 1$

$$[D^2u]_{\alpha;B_{\tau R}} \le \theta [D^2u]_{\alpha;R} + \cdots$$

$$\tau = \frac{1}{2} \qquad [D^2u]_{\alpha;B_{\frac{1}{2}R}} \le \{|f|_{\alpha;\Omega} + |u|_{0;\Omega}\}$$

Theorem 4.3 (Schauder interior estimate). Assume's coefficients satisfy (4.4) (4.5). $u \in C^{2,\alpha}(\Omega)$ $(0<\alpha<1)$ is the solution of (4.3)

Then for every $\Omega' \subset\subset \Omega$. we have:

$$|u|_{2,\alpha;\Omega'} \le C\left\{\frac{1}{\lambda}|f|_{0;\Omega} + |u|_{0;\Omega}\right\}$$

$$C = C\left(n, \alpha, \frac{\Lambda}{\lambda}, \Lambda_\alpha, \operatorname{dist}(\Omega', \partial\Omega)\right).$$

Proof: Let $\lambda = 1$. R_0 selected as Lemma 4.2

$\bar{R}_0 = \min\{R_0, \frac{1}{2}\operatorname{dist}(\Omega', \partial\Omega)\}$. For every $x_0 \in \Omega'$, $0 < R \le \bar{R}_0$

$B_R = B_R(x_0)$ $\left([u]_{2,\alpha;B_{\bar{R}_0}}\right)$. $u \in C^{2,\alpha}$

$\zeta(x)$ is a set-off function on B_R s.t

$$\begin{cases} \zeta \in C_0^\infty(\mathbb{R}^n),\ \zeta(x) = 1 \text{ when } x \in B_{\tau R} \\ \boxed{[D^k\zeta]_0 + (1-\tau)R^\alpha[D^k\zeta]_\alpha \le \frac{C(n,k)}{(1-\tau)^k R^k}} \end{cases} \qquad k=0.$$

$0 < \tau < 1$.

$0 \le \zeta \le 1$.

$$\frac{2}{(1-\tau)R}$$

$D\zeta \equiv 0$ $\qquad \boxed{|D\zeta| \le \frac{2}{(1-\tau)R}}$

$$|D\zeta|_0 \le \frac{1}{(1-\tau)R}.$$

$$[D^k\zeta]_0 + (1-\tau)R^\alpha [D^k\zeta]_\alpha \le \frac{C(n,k)}{(1-\tau)^k R^k}$$

$|D^k\rho| \le C.$

$\rho = \{$

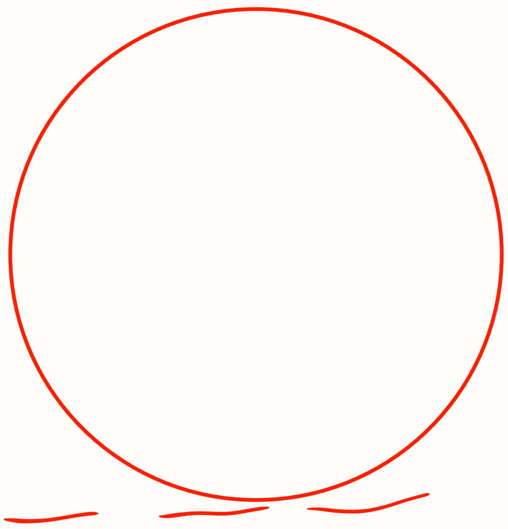
$B_{\tau R} : \zeta \equiv 1.$

$\overline{\operatorname{supp}}\, u \subset B_{(1+\varepsilon)R}.$

$h(x) = \chi_{B_{\tau R}}(x)$

$|D^k \rho_\varepsilon| \le \frac{C}{\varepsilon^k}.$

$\rho(x)$



$$D^k\zeta = h * \rho = h * D^k\rho.$$

$v = \zeta u.$ $\quad v \in C_0^{2,\alpha}(B_R).$

$$Lv = -a^{ij}D_{ij}(\zeta u) + b^i D_i(\zeta u) + c\zeta u.$$

$$= \zeta f + [-a^{ij}D_{ij}\zeta + b^i D_i \zeta]u - 2a^{ij}D_i\zeta D_j u = \hat f.$$

$$[D^2 v]_{\alpha;B_R} \le C[\hat f]_\alpha.$$

① ② ③ ?

$$\le C\left\{ [\zeta f]_\alpha + \left[[-a^{ij}D_{ij}\zeta + b^i D_i\zeta] u \right]_\alpha + [2a^{ij}D_i\zeta D_j u]_\alpha \right\}.$$

① $\le |\zeta|_0 [f]_\alpha + [\zeta]_\alpha |f|_0$

$$\le [f]_{\alpha;B_R} + \frac{1}{(1-\tau)^\alpha R^\alpha}[f]_{0;B_R}.$$

$$[D_i\zeta]_\alpha \le \frac{C}{[R(1-\tau)}$$

② $\le [a^{ij}D_{ij}\zeta]_\alpha |u|_0 + |a^{ij}D_{ij}\zeta|_0 [u]_\alpha + [b^i D_i\zeta]_\alpha |u|_0$

$$+ |b^i D_i\zeta|_0 [u]_\alpha .$$

$$[a^{ij}D_{ij}\zeta]_\alpha \le \frac{C}{[(1-\tau)R]^{2+\alpha}}$$

$$|a^{ij}D_{ij}\zeta|_0 \le \frac{C}{(1-\tau)^2R^2}$$

$$[u]_\alpha \le \varepsilon [D^2u]_{\alpha;B_R} + C_\varepsilon |u|_0 .$$

$$|a^{ij}D_{ij}\zeta|_0 [u]_\alpha \le \frac{C}{(1-\tau)^2R^2}\varepsilon [D^2u]_{\alpha;B_R} + \frac{C_\varepsilon}{(1-\tau)^2R^2}|u|_0$$

$$\le \frac{C_\varepsilon}{[(1-\tau)R]^{2+\alpha}}|u|_0$$

$$\frac{1}{(1-\tau)^2R^2} \le \frac{1}{[(1-\tau)R]^{2+\alpha}}$$

$D_j u]_\alpha .$

$D_j u]_\alpha + [a^{ij}D_i\zeta]_\alpha |D_j u|_0 .$

③ $= [D_i\zeta D_j u]_\alpha$

$$\le |D_i\xi|_0 [D_j u]_\alpha + [D_i\xi]_\alpha |D_j u|_0$$

$$\frac{1}{)R}\left[\varepsilon[D^2u]_\alpha + C_\varepsilon |u|_0\right] \le \frac{1}{[(1-\tau)R]^{1+\alpha}}\left[\varepsilon[D^2u] + C_\varepsilon |u|_0\right].$$

$< 2+\alpha$

$$[D^2 v]_{\alpha;B_R} \le C\left\{\frac{1}{(1-\tau)^\alpha R^\alpha}[f]_{0;B_R} + [f]_{\alpha;B_R} + \varepsilon[D^2u]_{\alpha;B_R} + \frac{C_\varepsilon}{[(1-\tau)R]^{2+\alpha}}|u|_{0;B_R}\right\}.$$

$R\downarrow$

ε

$\frac{1}{R}\uparrow$

$$[D^2u]_{\alpha;B_{\tau R}} \le C\left\{\frac{1}{(1-\tau)^\alpha R^\alpha}[f]_{0;B_R} + [f]_{\alpha;B_R} + \varepsilon[D^2u]_{\alpha;B_R} + \frac{C_\varepsilon}{[(1-\tau)R]^{2+\alpha}}|u|_0\right.$$

$$\varphi(s) = [D^2u]_{\alpha;B_s}.$$

$$\tau R = s,\quad R = t \qquad 0\le s<t\le R_0.$$

$$\varphi(s) \le C\left\{ \varepsilon\varphi(t) + [f]_{\alpha;B_{\bar R_0}} + \frac{1}{(t-s)^\alpha}[f]_{0;B_{\bar R_0}} + \frac{C_\varepsilon}{(t-s)^{2+\alpha}}|u|_0 \right\}.$$

$$C\varepsilon = \frac{1}{2}.$$

$$[D^2u]_{\alpha;B_\rho} \le C\left\{ [f]_{\alpha;B_{\bar R_0}} + \frac{1}{(R-\rho)^\alpha}[f]_{0;B_{\bar R_0}} + \frac{C}{(R-\rho)^{2+\alpha}}|u|_0 \right\}.$$

$$R = \bar R_0, \quad \rho = \frac{\bar R_0}{2}$$

$$u \in C^{2,\alpha}(B_{\bar R_0}).$$

$$|u|_{2,\alpha;B_{\frac{\bar R_0}{2}}} \le C\{|f|_{\alpha;\Omega} + |u|_{0;\Omega}\}.$$

Ω'

$\Omega' \subset\subset \Omega$.

Remark. $a^{ij}, b^i, c, f \in C^{k,\alpha}(\Omega)$.

$u \in C^{2,\alpha}(\Omega)$ is a solution of (4.3).

$u \in C^{k+2,\alpha}(\Omega)$.

$\forall\ \Omega' \subset\subset \Omega'' \subset\subset \Omega$

$$|u|_{k+2,\alpha;\Omega'} \le C\left(|f|_{k,\alpha;\Omega''} + |u|_{0;\Omega''}\right).$$