Lecture on 2D Vargant-Kazhikhov model. 4

$\mathbb{T}^2$. ○, $\mathbb{R}^2$.

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u) + \nabla p = \mu \Delta u + \nabla(2\mu + \lambda(\rho))\nabla\cdot u. \end{cases}$$

$\boxed{\mu = 1}$ $\mu = c > 0$ Huang 2016

$C^\alpha \leftarrow C^2$

2D Compressible

global regularity

$\beta > 0$

$\lambda(\rho) = \rho^\beta, \quad P(\rho) = \rho^\gamma.$ $\mu = c > 0$ μ $\lambda = 0$ $\lambda = c$

$\lambda = \rho^\beta$ $\rho = 0$ local $\beta > 3$ $\beta > \frac{4}{3}$ $\|\rho\|_{L^\infty}$

$\lambda =$ λ.

$\theta = \theta(\rho), \quad \theta'(\rho) = \frac{2\mu + \lambda(\rho)}{\rho}, \quad \theta \gtrsim \rho^\beta.$

$D_t\theta + D_t\xi + p + m + B = 0,$

Δ

$\xi = (\Delta^{-1})\nabla\cdot(\rho u),$

$m = [u^i, R_iR_j](\rho u^j),$

$B = (\bar{P} - \bar{P}).$

$\alpha > 0$ strong

$2 - \frac{4}{\alpha} = 1 > 0$ C^α $\alpha < 0$ $\alpha > 0$

$R_T = \|\rho\|_{L^\infty_T L^\infty}$,

Zlotnik.

$W^{2,4}$. $\|m\|$ $W^{2,3}$ L^2 weak

H^2 $2 - \frac{4}{p} > 0$ $p > \frac{n}{2}$ α

$\rho^\beta \lesssim \theta(\rho) \lesssim R_T^\eta,$

$W^{k,p} \hookrightarrow C^\alpha$.

$R_T^\beta \lesssim R_T^\eta, \quad \beta - \eta > 0, \Rightarrow R_T \lesssim C.$

$\alpha = \boxed{k - \cancel{\frac{n}{p}}\cdot\frac{n}{p}}$

$\oplus R_T$ $\alpha > 0$ L^∞ L^2

$m, \xi, B \lesssim A_1 \oplus A_2 \oplus A_3 \oplus \cancel{\varphi\alpha} \lesssim R_T.$

$\alpha = 0 - \frac{n}{2}$

$A_i \oplus \varphi\alpha \lesssim R_T.$ $\alpha = 0$

$L^\infty \to C^\infty$

Δ^{-1}

$\|\nabla u\|_{L^\varphi} \lesssim \varphi\alpha(\|F\|_{H^1} + \|w\|_{H^1}) + R_T^{\frac{3r+4\beta}{4}},$ A_3

$\lesssim \|\rho \dot{u}\|_{L^2} \sim A_2.$

$W^{c,p} \hookrightarrow L^\infty$

$\frac{d}{dt}A_1^2 + A_2^2 = I_1, \cdots, I_7.$

$p > 3$ $\alpha > 0$.

Weak Solution $\alpha = 0$ $\alpha = 0$ $\alpha = 0$

BMO $L^\infty \to C^0$

L^∞ strong

① $A_1, A_2, A_3 \lesssim R_T$.

$\|\rho\|_{L^\infty L^q} \lesssim C(T)$.

$\|\rho\| \; R_T = \|\rho\|_{L^\infty}$

$\|\rho\|_{L^q}$

② $m, \xi, B \lesssim A_i R_T \lesssim R_T$.

Δ

$A_1 =: \int \mu |w|^2 + \int \frac{F^2}{2\mu+\lambda}$,

$A_2 =: \int \rho |\dot u|^2$

$A_3 =: \int \mu |w|^2 + \int (2\mu+\lambda)|\nabla\cdot u|^2$

$\varphi_\alpha = 1 + \|\frac{\rho}{2\mu+\lambda}\|_{L^2} + R_T^{\alpha\beta} A_1^{\frac{1}{2}}$.

Estimate of A_1, A_2, A_3:

$$\frac{d}{dt} A_1^2 + A_2^2 = \overset{I_1}{-2\mu\int |w|^2 \nabla\cdot u} + \overset{I_2}{4\int F \nabla u_1 \cdot \nabla^\perp u_2},$$

$$I_3 \quad -2\gamma\int \frac{\rho}{2\mu+\lambda} F\nabla\cdot u + \int \frac{\lambda(\rho-\bar\rho)}{(2\mu+\lambda)^2} F\nabla\cdot u \quad I_\varphi$$

$$I_5 \quad -\int \frac{(\beta-1)\lambda-2\mu}{(2\mu+\lambda)^2} F^2\nabla\cdot u - 2\int F|\nabla\cdot u|^2 \quad I_6$$

$A_3 \|F\|_{L^2}$

$$I_7 \quad -2(\gamma-1)\int \frac{F}{2\mu+\lambda}\int \rho\nabla\cdot u \lesssim \varphi_\alpha(\|F\|_{H^1} + \|w\|_{H^1}) A_3 + A_3^2$$

$\nabla\cdot w \Rightarrow$

$$I_1 \le \int |w|^2 (\nabla\cdot u) \le C\|w\|_{L^4}^2 \|\nabla\cdot u\|_{L^2} \le \underbrace{\|w\|_{L^2}}_{\lesssim A_1} \underbrace{\|\nabla w\|_{L^2}}_{\lesssim \|w\|_{H^1}} \underbrace{\|\nabla\cdot u\|_{L^2}}_{\lesssim A_3}$$

$$\lesssim C\underbrace{A_1}_{\le \varphi_\alpha} \|w\|_{H^1} A_3 \lesssim \varphi_\alpha \|w\|_{H^1} A_3.$$

$1-\frac{2}{2}$ $\quad$ 0

$2D$

$H^1 \hookrightarrow BMO$

$L^\infty \leftrightarrow L^1$

$$I_2 \lesssim \int F \nabla u_1 \cdot \nabla^\perp u_2 \lesssim \|F\|_{BMO} \|\nabla u_1 \cdot \nabla^\perp u_2\|_{H^1} <$$

$BMO \leftrightarrow H^1$

$\lesssim \|\nabla F\|_{L^2} \|\nabla u\|_{L^2}^2$

$L^\infty \leftarrow L^1$

$\lesssim \varphi_\alpha \|F\|_{H^1} A_3$.

$F\cdot w_{\partial\Omega} = 0 \quad F\times n|_{\partial\Omega} = 0$

I_3, I_4, I_5, I_6

φ_α

$$I_5 \lesssim C\int \underbrace{\frac{F^2}{2\mu+\lambda}}_{L^2} \underbrace{|\nabla\cdot u|}_{L^2} \lesssim C(\alpha)\varphi_\alpha \|F\|_{H^1} A_3.$$

$\|F\|_{L^p} \lesssim \|F\|_{L^2}^{\alpha} \|\nabla F\|_{L^2}^{1-\alpha} + \|F\|_{L^2}$

F^2

$\|F\|_{L^4} \leftarrow \|F^2\|_{L^2} \lesssim \|F\|_{L^{2+}} \|F\|_{L^{\infty-}} \lesssim \|\frac{F}{\sqrt{2\mu+\lambda}}\|_{L^2}^{1-\alpha} \|F\|_{L^{2+\frac{2}{\alpha}}}^{\alpha}$

$\lesssim A_1^{1-\alpha} \|F\|_{L^2}^{\alpha} \|F\|_{H^1}$

$\|F\|_{L^2}$ $\quad$ $\|\nabla F\|_{L^2}$ $\quad$ $\|F\|_{H^1}$

$\lesssim C(\alpha)\underbrace{A_1 R_T^{\frac{\alpha}{2}}}_{\varphi_\alpha}\|F\|_{H^1}$

$\|\nabla F\|_{L^2}\lesssim\|\rho\dot u\|_{L^2}$

$I_5\lesssim C(\alpha)\varphi_\alpha\|F\|_{H^1}A_3$ $\quad\tilde F$

$\frac{F}{\sqrt{2\mu+\lambda}}\sim\frac{F}{\frac{\beta}{2}}\lesssim A_1$

$\|F\|_{H^1}\lesssim$

$\|\nabla F\|_{L^2}+\|F-\bar F\|_{L^2}\lesssim\|\rho\dot u\|_{L^2}$

$F^{\alpha}\lesssim\rho^{\frac{\beta}{2}}$ $\quad +\bar F$

$R_T^{\frac{\alpha}{2}}A_1$. $\quad\alpha$.

$$I_3+I_4\lesssim\int\frac{|P+1|}{2\mu+\lambda(\rho)}F|\nabla\cdot u|.$$

$$\lesssim\left\|\frac{P+1}{(2\mu+\lambda)^{\frac{1}{2}}}\right\|_{L^{2+\frac{\beta}{\gamma}}}\|F\|_{L^{2+\frac{4\gamma}{\beta}}}\|(2\mu+\lambda)^{\frac{1}{2}}\nabla\cdot u\|_{L^2}$$

Δ?

$$\lesssim c(\alpha)\varphi_\alpha\|F\|_{H^1}A_3$$

$$I_7=\int\frac{F}{2\mu+\lambda}\int P\nabla\cdot u.\quad\nearrow 0$$

$\int\nabla\cdot u$

$\int\frac{P}{2\mu+\lambda}\cdot\rho^{\gamma-\beta}$

$$\lesssim\int\left(\nabla\cdot u-\frac{P}{2\mu+\lambda}\right)\int(F-(2\mu+\lambda)\nabla\cdot u)\nabla\cdot u$$

$$\lesssim A_3\|F\|_{L^2}+A_3^2$$

$\|\nabla u\|\,\|F\|\qquad\|\nabla u\|^2$

$\int\rho^\beta\ A_3$ $\quad(2\mu+\lambda)^{\frac{1}{2}}\ (2\mu+\lambda)^{\frac{1}{2}}\nabla\cdot u$

$$\frac{d}{dt}A_1^2+2A_2^2\lesssim C\alpha(\|F\|_{H^1}+\|\omega\|_{H^1})A_3+A_3^2$$

$\mathrm{II}^2\to 1$

$$\lesssim C\alpha(\|\rho\dot u\|_{L^2}+\|F-\bar F\|_{L^2}+|\bar F|)A_3+A_3^2$$

$|\bar F|=\left|\int(2\mu+\lambda)\nabla\cdot u\right|+\int P$

$$\lesssim C\alpha\left(R_T^{\frac{1}{2}}A_2+\|P\|_{L^\beta}^{\frac{\beta}{2}}A_3\right)A_3+A_3^2$$

$(\|P\|_{L^\beta}^{\frac{\beta}{2}}$ $\quad=\|(2\mu+\lambda)^{\frac{1}{2}}\|_{L^2}\ \rho^{\frac{1}{2}}$

$\|(2\mu+\lambda)^{\frac{1}{2}}\nabla\cdot u\|_{L^2}+\int P$

$$\lesssim A_2\cdot C\alpha R_T^{\frac{1}{2}}\varphi_\alpha A_3+C\alpha R_T^{\frac{1}{2}}\varphi_\alpha A_3$$

$$\cdot R_T^{-\frac{1}{2}}\|P\|_{L^\beta}^{\frac{\beta}{2}}A_3+CA_3^2$$

A_3

$\int(\rho^{\beta-\gamma})^2$

$$\lesssim A_2^2+C\alpha R_T\varphi_\alpha^2A_3^2+R_T^{-1}\|P\|_{L^\beta}^{\beta}A_3^2+CA_3^2$$

$$\frac{d}{dt}A_1^2+A_2^2\lesssim C\alpha R_T\left(\varphi_\alpha^2+R_T^{-2}\|P\|_{L^\beta}^{\beta}\right)A_3^2$$

$\varphi_\alpha=1+\left\|\frac{P}{2\mu+\lambda}\right\|_{L^2}+R_T^{\frac{\alpha\beta}{2}}A_1^{\frac{1}{2}}$

$R_T^{\frac{\gamma-2\beta}{2}}$

$$\lesssim C\alpha R_T\left(1+R_T^{\gamma-2\beta}+R_T^{\alpha\beta}A_1+R_T^{\beta-\gamma-2}\right)A_3^2$$

$$\lesssim C\alpha R_T\left(R_T^{k}+R_T^{\alpha\beta}A_1^2\right)A_3^2,\quad k=\max\{0,\gamma-2\beta,\beta-\gamma-2\}$$

$(e+A_1^2) \lesssim R_T^k \lesssim R_T^{\alpha\beta}$

$A_1, A_2, A_3 \leq R_T$

$A_3^2 = \int |\nabla w|^2 + \int (2M+\lambda) |D\cdot u|^2$

energy inequality

$(e+A_1^2)$

$\int A_3^2 \lesssim 1$

$$\frac{\frac{d}{dt} A_1^2}{(e+A_1^2)} + \frac{A_2^2}{(e+A_1^2)} \lesssim C_\alpha R_T (R_T^k + R_T^{\alpha\beta}) A_3^2$$

$$\frac{d}{dt} \log(e+A_1^2) + \frac{A_2^2}{e+A_1^2} \lesssim C_\alpha R^{1+k+\alpha\beta} A_3^2 \qquad \int A_3^2 \lesssim 1.$$

$\beta \cdot r -$

$$\log(e+A_1^2) + \int \frac{A_2^2}{e+A_1^2} \lesssim C_\alpha R_T^{\cancel{1+k+\alpha\beta}}$$

$+A_3^2$

d

$$\log(e+A_3^2) \lesssim \log(e+A_1^2) + \log R_T \leq R_T^{1+k+\alpha\beta}$$

R_T^0

A_1, A_3

$A_3 \sim |\nabla \times u|, |D\cdot u|$

$R_T^{2r-\beta}$ $\quad \int P \rho^r$.

$A_1 \sim |\nabla \times u|, |f|$

$$A_1 \lesssim A_3 + R_T^{2r-\beta}$$

$\log(e+x+y) \leq \log(e+x) + \log(e+y)$

$$\log(e+A_1) \lesssim \log(e+A_3) + \log R_T$$

$C \qquad R_T^0$

$$\log(e + A_1^2 + A_3^2) + \int \frac{A_2^2}{e+A_1^2} \lesssim C_\alpha R_T^{1+\alpha+\alpha\beta}.$$

$$A_1, A_2, A_3 \leq R_T$$

estimate of $\|\rho\|_{L^q}$ - $\lesssim$

$\beta > \frac{4}{3}$. $\|\rho\|_{L^\infty_T L^\infty} \leq C(T)$ VK global exists.

$\beta > \cancel{\frac{1}{2}}\ \gamma$ $\qquad \|\rho\|_{L^\infty_T L^\infty} \leq C$

$\|\rho\|_{L^\infty_T L^q} \leq C(T)$, $q = 2\beta\gamma + 1$

kazhikhov $\quad \cancel{\|\rho\|}\ \forall q \geq 4$.

$D_t\theta + D_t\xi + p + m + B = 0$.

$D_t(\underbrace{\theta + \xi}_{f}) + p + m + B = 0$

$D_t f$. $\rho f^{2\gamma - 1}$

$\rho^{2\beta\gamma+1}$

$\rho f^{2\gamma} + C \gtrsim \int \rho^{2\beta\gamma+1}$

$\rho f^{2\gamma}$

$f \quad \underset{\Delta}{\theta} + \xi$, ρ^β

$f = (\theta + \xi)_+$

$\rho \cdot \rho^{2\gamma \cdot \beta} \to \rho^{2\beta\gamma+1}$

$\rho\theta^{2\gamma}$

$f = \theta + \xi \leftarrow \theta \lesssim |\theta + \xi| + |\xi|$

$$\int \rho^{2\beta\gamma+1} \lesssim \int \rho + \int \rho f^{2\gamma} + \int \rho |\xi|^{2\gamma}.$$

Pointl

$$\frac{d}{dt} \rho f^{2\gamma} \lesssim (1 + \int \rho f^{2\gamma} + \int \rho^{2\beta+1})(1 + \|\nabla u\|^2_{L^2})$$

L^2 c

$$\int \rho^{2\beta\gamma+1} \lesssim \int_{\rho\leq c} \rho^{2\beta\gamma+1} + \int_{\rho\geq c} \rho(\rho^\beta)^{2\gamma} \qquad \rho^\beta \lesssim \theta \lesssim |\theta + \xi| + |\xi|$$

f

$$\lesssim \int_{\rho\leq c} \rho + \int_{\rho\geq c} \rho f^{2\gamma} + \int_{\rho\geq c} \rho |\xi|^{2\gamma}.$$

$$\lesssim 1 + \int \rho f^{2\gamma} + \int \rho |\xi|^{2\gamma}$$

$\xi = (\Delta^{-1})(\nabla \cdot (\rho u))$. $\lesssim \|\rho u\|$, ∇^{-1}

$\rho u \leftarrow \rho^{\frac{1}{2}} u \leq c$

$\xi \leq \rho$.

$$\int \rho |\xi|^{2\gamma} \lesssim \|\rho\|_{L^{\frac{q}{q-2\gamma}}} \|\xi\|^{2\gamma}_{L^q} \lesssim \|\rho\|_{L^{\frac{q}{q-2\gamma}}} \|\nabla\xi\|^{2\gamma}_{L^{\frac{2q}{q+2}}} \lesssim \|\rho\|_{L^{\frac{q}{q-2\gamma}}} \|\rho u\|_{L^{\frac{2q}{q+2}}}.$$

$$\lesssim \|\rho\|_{L^{\frac{q}{q-2r}}} \|\rho\|^{r}_{L^{\frac{q}{2}}} \|\rho^{\frac{1}{2}} u\|^{2r}_{L^2} \lesssim 1.$$

$$\frac{q}{2} = 2\beta r + 1 \Rightarrow q = 2(2\beta r+1) \Rightarrow \frac{q}{q-2r} = \frac{2\beta r+1}{(2\beta-1)r+1} \in (1, 2\beta r+1)$$

$$\lesssim 1 + \int \rho f^{2r} + \underbrace{\|\rho\|_{L^{\frac{2\beta r+1}{(2\beta-1)r+1}}}}_{} \|\rho\|^{r}_{L^{2\beta r+1}}$$

$C(1+\|\rho\|_{L^{2\beta r+1}})$.

$$\le C_\varepsilon + \int \rho f^{2r} + \varepsilon \int \rho^{2\beta r+1}.$$

$$\int \rho^{2\beta r+1} \lesssim C_\varepsilon + \int \rho f^{2r}$$

$\triangle$. ~~$\|m\|_{L^\infty}$~~ $\|m\|_{L^q}$. ρu^2.

$[u^i, R_i R_j](\rho u^j)$

$$\frac{d}{dt}\int(\rho f^{2r}) \lesssim C\int \rho f^{2r-1}|m|$$

$$\lesssim \|\rho\|^{\frac{1}{2r}}_{L^{2\beta r+1}} \|\rho^{\frac{1}{2r}} f\|^{2r-1}_{L^{2r}} \|m\|_{L^q} \qquad q = \frac{2\beta r+1}{\beta}$$

$\|\rho^{\frac{1}{2}} u\|_{L}$

$$\lesssim C\|\rho\|^{\frac{1}{2r}}_{L^{2\beta r+1}} \|\rho^{\frac{1}{2r}} f\|^{2r-1}_{L^{2r}} \|\nabla u\|_{L^2} \|\rho u\|_{L^q} \leftarrow C(1+\|\rho\|_{L^{2\beta r+1}})$$

$$\lesssim C\left(1+\int \rho f^{2r} + \int \rho^{2\beta r+1}\right)(1+\|\nabla u\|^2_{L^2}).$$

$\in 1 + \int \rho f^{2r}.$

$$\int \rho f^{2r} \lesssim C(T)$$

$$\int \rho^{2\beta r+1} \lesssim C(T)$$

$\|m\|_{L^\infty}$ $\|\rho\|_{L^\infty}$

①

②