Global regularity of Kazhikhov model J.

$$D_t\theta + D_t\xi + p + m + B = 0,$$

$\theta = \theta(\rho)$, $\theta'(\rho) = \frac{2\mu+\lambda(\rho)}{\rho}$,

$\xi = \Delta^{-1}(\nabla\cdot(\rho u))$,

$m = [u^i, R_i R_j](\rho u^j)$.

↓ global regularity

$$\|\rho\|_{L^\infty_T L^\infty} = R_T \implies R_T \lesssim C \,/\, R_T \lesssim C(T).$$

↑ Zlotnik

$\|\xi\|_{L^\infty}$, $\|m\|_{L^\infty}$, $|B|$.

↑

$$\|m\|_{L^\infty} \lesssim \|\nabla u\|_{L^2} \oplus \|\nabla u\|_{L^p} \oplus \|\rho u\|_{L^q}.$$

← uniform case.

$\|\rho\|_{L^q}$ ← optimal case. (time-dependent)

↑

$$A_1^2 =: \int \mu|\omega|^2 + \frac{F^2}{|2\mu+\lambda|}.$$

$$A_2^2 = \int \rho|\dot{u}|^2,$$

$$A_3^2 = \int \mu|\omega|^2 + (2\mu+\lambda)|\nabla\cdot u|^2.$$

$$\varphi_\alpha = 1 + \left\|\frac{P}{2\mu+\lambda}\right\|_{L^2} + A_1 R_T^{\frac{\alpha\beta}{2}}$$

$$\|\nabla u\|_{L^p} \lesssim C(\alpha)\varphi_\alpha^{\frac{1}{2}}\left(R_T^{\frac{1}{2}}A_2 + \|\rho\|_{L^\beta}^{\frac{\beta}{2}}A_3\right)^{\frac{1}{2}} + CR_T^{\frac{3\gamma-4\beta}{4}}$$

$$\log(e+A_1^2+A_3^2) + \int_0^T \frac{A_2^2}{e+A_1^2} \leq C(\alpha)R_T^{1+k+\alpha\beta} \,/\, C(\alpha,T)R_T^{1+\alpha\beta},$$

$$\varphi_\alpha \lesssim C\left(1 + R_T^{\frac{\gamma-2\beta}{2}} + A_1 R_T^{\frac{\alpha\beta}{2}}\right) \lesssim CR_T^{\frac{k+\alpha\beta}{2}}(e+A_1).$$

$$C(T)\left(1 + A_1 R_T^{\frac{\alpha\beta}{2}}\right).$$

$$k = \max\{0, \gamma-2\beta, \beta-\gamma-2\}.$$

$$\|\rho\|_{L^{2\beta\gamma+1}} \lesssim C(T).$$

$|\vec{F}| = \int_{\mathbb{T}^2} (2\mu + \rho^\beta)\nabla\cdot u - \nabla P$

$= \int_{\mathbb{T}} \frac{\rho^{\frac{\beta}{2}}\sqrt{2\mu+\lambda}\,\nabla\cdot u}{(2\mu+\rho^\beta)\nabla\cdot u}$

$\leq \|\rho^{\frac{\beta}{2}}\|_{L^2} \|(2\mu+\lambda)^{\frac{1}{2}}\nabla\cdot u\|_{L^2}$

$\|\rho\|_{L^\beta}^{\frac{\beta}{2}} \quad A_3$

$$\|\nabla u\|_{L^\varphi} \leq C(\alpha)\varphi_\alpha^{\frac{1}{2}}\left(R_T^{\frac{1}{2}}A_2 + \|\rho\|_{L^\beta}^{\frac{\beta}{2}}A_3\right)^{\frac{1}{2}} + CR_T^{\frac{3r-\varphi\beta}{\varphi_1}}$$

$$\leq C(\alpha)\varphi_\alpha^{\frac{1}{2}}R_T^{\frac{1}{\varphi}}A_2^{\frac{1}{2}} + C(\alpha)\varphi_\alpha^{\frac{1}{2}}\|\rho\|_{L^\beta}^{\frac{\beta}{\varphi}}A_3^{\frac{1}{2}} + CR_T^{\frac{3r-\varphi\beta}{\varphi}}$$

$$\leq C(\alpha)R_T^{\frac{k}{\varphi}+\frac{\alpha\beta+1}{\varphi}}(e+A_1)^{\frac{1}{2}}\left(\frac{R_T^{-\varphi-k}A_2^2}{e+A_1^2}\right)^{\frac{1}{\varphi}}(e+A_1^2)^{\frac{1}{\varphi}}$$

$$+ C(\alpha)R_T^{\frac{k+\alpha\beta+\beta}{\varphi}}(e+A_1)^{\frac{1}{2}}(e+A_3)^{\frac{1}{2}} + CR_T^{\frac{3r-\varphi\beta}{\varphi}}$$

$$\leq C(\alpha)R_T^{\bar{c}}\left(\frac{R_T^{-\varphi-k}A_2^2}{e+A_1^2}\right)^{\frac{1}{\varphi}}(e+A_3) + CR_T^{\bar{c}}(e+A_3)$$

$\left(R_T^{-\varphi-k}\right.$

$(e+A_1) \leq (e+A_3)$

Uniform case:

Brezis–Wagner inequality:

$$\|\rho u\|_{L^q} \leq R_T \oplus \|\rho^{\frac{1}{2}}u\|_{L^2} \oplus \|u\|_{L^\infty}\left(\leq \|\nabla u\|_{L^2} \oplus \log(e+\|\nabla u\|_{L^\varphi})\right).$$

($R_T$: $\|\rho\|_{L^\infty}$)

$$\log(e+\|\nabla u\|_{L^\varphi}) \leq C\log(e+A_3^2) + \frac{1}{\varphi}\log\left(e+\frac{R_T^{-\varphi-k}A_2^2}{(e+A_1^2)(e+A_3)^\varphi} + \frac{1}{(e+A_3)^\varphi}\right).$$

$$\leq C(\alpha)R_T^{1+k+\alpha\beta} + C\frac{R_T^{-\varphi-k}A_2^2}{(e+A_1^2)(e+A_3)^6}$$

$$\|u\|_{L^\infty}^2 \leq C\|\nabla u\|_{L^2}^2\left(1+\log^{\frac{1}{2}}(e+\|\nabla u\|_{L^4})\right) + C.$$

$$\leq CA_3\left(1+C_\alpha R_T^{\frac{1+k+\alpha\beta}{2}} + \left(\frac{R_T^{-\varphi-k}A_2^2}{(e+A_1^2)(e+A_3)^6}\right)^{\frac{1}{2}}\right) + C.$$

$$\leq C(\alpha)R_T^{\frac{1+k+\alpha\beta}{2}}A_3 + C\left(\frac{R_T^{-\varphi-k}A_2^2}{(e+A_1^2)(e+A_3^4)}\right)^{\frac{1}{2}} + C$$

$$\|\rho u\|_{L^q} \leq CR_T^{1-\frac{1}{q}}\|\rho^{\frac{1}{2}}u\|_{L^2}^{\frac{2}{q}}\|u\|_{L^\infty}^{1-\frac{2}{q}}$$

$$\leq CR_T^{1-\frac{1}{q}}\left(C(\alpha)R_T^{\frac{1+k+\alpha\beta}{2}}A_3 + C\left(\frac{R_T^{-\varphi-k}A_2^2}{(e+A_1^2)(e+A_3^4)}\right)^{\frac{1}{2}} + C\right)^{(1-\frac{2}{q})}.$$

$$\leq C(\alpha)R_T^{\frac{3+\alpha\beta+k}{2}}A_3^{1-\frac{2}{q}} + C\left(\frac{R_T^{-1-k}A_2^2}{(e+A_1^2)(e+A_3)^\varphi}\right)^{\frac{1}{2}-\frac{1}{q}} + CR_T.$$

$$\|m\|_{L^\infty} \le C(q)\|\nabla u\|_{L^2}^{\frac{q-4}{q}}\|\nabla u\|_{L^4}^{\frac{4}{q}}\|\rho u\|_{L^q}$$

$$\lesssim C(q)\|\nabla u\|_{L^4}^{\frac{4}{q}} A_3^{\frac{q-4}{q}}\left(R_T^{\frac{3+\alpha\beta+k}{2}}A_3^{1-\frac{2}{q}}+\left(\frac{R_T^{-1-k}A_3^2}{(e+A_1^2)(e+A_3)^4}\right)^{\frac{1}{2}-\frac{1}{q}}+CR_T\right)$$

$$\lesssim C(q)\|\nabla u\|_{L^4}^{\frac{4}{q}} A_3^{2-\frac{6}{q}}R_T^{\frac{3+\alpha\beta+k}{2}} \quad ①$$

$$+C(q)\|\nabla u\|_{L^4}^{\frac{4}{q}}A_3^{1-\frac{4}{q}}\left(\frac{R_T^{-1-k}A_3^2}{(e+A_1^2)(e+A_3)^4}\right)^{\frac{1}{2}-\frac{1}{q}}+C(q)\|\nabla u\|_{L^4}^{\frac{4}{q}}A_3^{1-\frac{4}{q}}R_T.$$

$\lesssim$.

$\nabla\cdot u$ ?

$$\|\nabla u\|_{L^4} \quad C(\alpha)R_T^{\tilde{C}}\left(\frac{R_T^{-4k}A_2^2}{e+A_1^2}\right)^{\frac{1}{4}}(e+A_3)+C(\alpha)R_T^{\tilde{C}}(e+A_3).$$

$$① \quad C(q)\|\nabla u\|_{L^4}^{\frac{4}{q}}A_3^{2-\frac{6}{q}}R_T^{\frac{3+\alpha\beta+k}{2}}$$

$$\lesssim C(q,\alpha)A_3^{2-\frac{6}{q}}(e+A_3)^{\frac{4}{q}}\left(\frac{R_T^{-1-k}A_2^2}{e+A_1^2}\right)^{\frac{1}{q}}R_T^{\frac{3+k+\alpha\beta}{2}+\frac{4\tilde{C}}{q}}.$$

$$+C(q,\alpha)\|\nabla u\|_{L^4}^{\frac{4}{q}}\left(\frac{R_T^{-1-k}A_2^2}{e+A_1^2}\right)^{\frac{1}{2}-\frac{1}{q}}$$

$$+C(q)R_T(A_3+1)\|\nabla u\|_{L^4}^{\frac{4}{q}} = J_1, J_2, J_3, J_4.$$

$$J_1 =: C(q,\alpha)A_3^{2-\frac{6}{q}}(e+A_3)^{\frac{4}{q}}\left(\frac{R_T^{-1-k}A_2^2}{e+A_1^2}\right)^{\frac{1}{q}}R_T^{\frac{3+k+\alpha\beta}{2}+\frac{4\tilde{C}}{q}}.$$

$$\lesssim C(q,\alpha)\left(R_T^{\frac{3+k+\alpha\beta}{2}+\frac{4\tilde{C}}{q}}A_3^{2-\frac{6}{q}}\right)^{\frac{q}{q-3}}$$

$$+C(q,\alpha)(e+A_3^2)+\frac{R_T^{-1-k}A_v^2}{e+A_1^2}.$$

$$\lesssim C(q,\alpha)\, C(1+A_3^2)R_T^{\hat{k}}+\frac{R_T^{-1-k}A_2^2}{e+A_1^2}$$

$$\hat{k}=\left(\frac{3+k+\alpha\beta}{2}+\frac{4\tilde{C}}{q}\right)\frac{q}{q-3}$$

$$J_2, J_3 \lesssim CR_T\|\nabla u\|_{L^4}^{\frac{8}{q}}+\frac{R_T^{-1-k}A_v^2}{e+A_1^2}+C(q,\alpha)R_T+C(q)R_TA_3^2.$$

$$\lesssim C(\alpha,q)R_T^{1+\frac{16\tilde{C}}{q}}+C(\alpha,q)R_T^{1+\frac{16\tilde{C}}{q}}A_3^2+\frac{CR_T^{-1-k}A_v^2}{e+A_1^2}$$

$$\|m\|_{L^\infty} \le C(q,\alpha)R_T^{1+\frac{16\tilde{C}}{q}}+C(q,\alpha)R_T^{1+\frac{16\tilde{C}}{q}+\left(\frac{3+k+\alpha\beta}{2}+\frac{4\tilde{C}}{q}\right)}A_3^2+\left(\frac{R_T^{-1-k}A_v^2}{e+A_v^2}\right)$$

$\forall \varepsilon > 0$, $q_1 \to \infty$, $\downarrow$

$\alpha \to 0$ $\quad \|m\|_{L^\infty} \leq \dfrac{C(\varepsilon) R_T^{-1+} A_2^2}{e + A_1^2} + C(\varepsilon) A_3^2 R_T^{\frac{3+k}{2}+\varepsilon} + C(\varepsilon) R_T^{1+\varepsilon}$.

$q(\varepsilon)$

$\alpha(\varepsilon)$

$\|m\|_{L^\infty}$. ( uniform case).

$\|m\|_{L^\infty}$. ( optimal case. $\beta > \frac{4}{3}$ $\quad \|\rho\|_{L^{2\beta\gamma+1}} \leq C(T)$ )

$\|m\|_{L^\infty} \leq \|\nabla u\|_{L^2} \oplus \|\nabla u\|_{L^q} \oplus \|\rho u\|_{L^q}$ $\left( \leq \|\rho u\|_{L^{2+\delta}} \oplus \|\rho u\|_{L^r} \right)$,

$\leq$

① $\|\rho u\|_{L^{2+\delta}}$

$D_t(\rho u) + \nabla p = \mu \Delta u + \nabla(\mu + \lambda(\rho)) \nabla \cdot u$ test: $|u|^\delta u$.

to get:

$$\frac{1}{2+\delta} \frac{d}{dt} \int \rho |u|^{2+\delta} + \int |u|^\delta \left( \mu |\nabla u|^2 + (\mu + \lambda(\rho)) |\nabla \cdot v|^2 \right) + \delta \mu \int |u|^\delta |\nabla |u||^2$$

$$= -\delta \int (\mu + \lambda(\rho)) (\nabla \cdot u) |u|^{\delta-1} u \cdot \nabla |u| - \frac{1}{2+\delta} \int P \nabla \cdot (|u|^\delta u). \quad R_1.$$

$$\leq \delta \int (\mu + \lambda(\rho)) |\nabla \cdot u| \cdot |u|^\delta |\nabla u| - \frac{1}{2+\delta} \int \rho^\gamma |u|^\delta |\nabla u|.$$

$\|\rho\|_{L^\infty} \leq C(T)$

$$\leq \frac{1}{2} \int (\mu + \lambda(\rho)) |\nabla \cdot u|^2 |u|^\delta + \frac{\delta^2}{2} \int (\mu + \lambda(\rho)) |\nabla u|^2 |u|^\delta + \varepsilon \int \mu |\nabla u|^2 |u|^\delta + C \int \rho |u|^{2+\delta} + \int \rho^{(2+\delta)\gamma - \frac{\delta}{2}}$$

$\int \rho |u|^{2+\delta} \leq C(T)$, $\quad \delta = C(\mu) R_T^{-\frac{\beta}{2}}$.

consequently, we have that for any $q > 4$,

$$\|\rho u\|_{L^q} \leq C \|\rho u\|_{L^{2+\delta}}^{\frac{2}{q}} \|\rho u\|_{L^r}^{(1-\frac{2}{q})}, \quad r \quad r.$$

$$\leq C R_T^{\frac{1+\delta}{2+\delta} \cdot \frac{2}{q}} \|\rho^{\frac{1}{2+\delta}} u\|_{L^{2+\delta}}^{\frac{2}{q}} R_T^{1-\frac{2}{q}} \left( r^{\frac{1}{3}} \|u\|_{H^1} \right)^{1-\frac{2}{q}}.$$

$$\leq C R_T^{1-\frac{2}{2+\delta}q}\, r^{(\frac{1}{2}-\frac{1}{q})} \|u\|_{H^1}^{1-\frac{2}{q}} \qquad r = \frac{(q-2)(2+\delta)}{\delta} \sim \frac{1}{-\delta}$$

$$\lesssim C R_T^{1-\frac{2}{(2+\delta)q}} R_T^{\frac{\beta}{2}(\frac{1}{2}-\frac{1}{q})} \|u\|_{H^1}^{1-\frac{2}{q}}$$

$$\lesssim C R_T^{1+\beta(\frac{1}{4}-\frac{1}{2q})} \|u\|_{H^1}^{1-\frac{2}{q}}$$

$$\|m\|_{L^\infty} \lesssim C(q) \|\nabla u\|_{L^2}^{\frac{q-\varphi}{q}} \|\nabla u\|_{L^\varphi}^{\frac{\varphi}{q}} \|\rho u\|_{L^q} + C(q)\|F\|_{L^q}$$

Estimate: for $\|\nabla u\|_{L^\varphi}$: $\quad \|\nabla u\|_{L^\varphi} \lesssim C R_T^{\bar{c}} (e+A_3)\left(\frac{R_T^{-4-k} A_2^2}{e+A_1^2}\right)^{\frac{1}{\varphi}} + C R_T^{\bar{c}}(e+A_3)$

Estimate for $\|\rho u\|_{L^q} \lesssim C R_T^{1+\beta(\frac{1}{4}-\frac{1}{2q})} \|u\|_{H^1}^{1-\frac{2}{q}}$

$\downarrow$ $\|m\|_{L^\infty}$

$$\|m\|_{L^\infty} \lesssim C(q) \|\nabla u\|_{L^2}^{\frac{q-\varphi}{q}} \|\nabla u\|_{L^\varphi}^{\frac{\varphi}{q}} \|\rho u\|_{L^q}$$

$$\lesssim C(q,T) A_3^{\frac{q-\varphi}{q}} \cdot R_T^{1+\frac{\beta(q-2)}{4q}} (1+A_3)^{1-\frac{2}{q}} \cdot \|\nabla u\|_{L^\varphi}^{\frac{\varphi}{q}}$$

$$\lesssim C(q,T) R_T^{1+\frac{k(q-2)}{4q}} (1+A_3^{2-\frac{6}{q}}) \left(R_T^{\bar{c}}(e+A_3)\left(\frac{R_T^{-4k} A_2^2}{e+A_1^2}\right)^{\frac{1}{\varphi}} + R_T^{\bar{c}}(e+A_3)\right)^{\frac{\varphi}{q}}$$

$$\lesssim C(q,T)\left(R_T^{1+\frac{\beta(q-2)}{4q}+\frac{4\bar{c}}{q}} (1+A_3^{2-\frac{2}{q}}) \left(\frac{R_T^{-4k} A_2^2}{e+A_1^2}\right)^{\frac{1}{q}} + R_T^{1+\frac{\beta(q-2)}{4q}+\frac{4\bar{c}}{q}} (1+A_3^{2-\frac{2}{q}})\right)$$

$$\left[R_T^{\frac{q}{q-1}+\frac{\beta(q-2)}{4(q-1)}+\frac{4\bar{c}}{q-1}} (1+A_3)^2\right]^{1-\frac{1}{q}} \left(\frac{A_2^2}{e+A_1^2}\right)^{\frac{1}{q}}$$

$$\lesssim C(q,T) R_T^{1+\frac{\beta}{4}+\frac{8c}{q-1}} (1+A_3^2)^2 + \frac{A_2^2}{e+A_1^2}$$

$$\|\xi\|_{L^\infty} \le C\|\nabla\xi\|_{L^2}\log^{\frac{1}{2}}(e+\|\nabla\xi\|_{L^q}) + C\|\xi\|_{L^2} + C$$

$$\le C\|\rho u\|_{L^2}\log^{\frac{1}{2}}(e+\|\rho u\|_{L^q}) + C\|\rho u\|_{L^{\frac{2r}{r+1}}} + C$$

$$\le CR_T^{\frac{1}{2}}\|\rho^{\frac{1}{2}}u\|_{L^2}\log^{\frac{1}{2}}(e+R_T(1+\|\nabla u\|_{L^2})) + C\|\rho\|_{L^r}^{\frac{1}{2}}\|\rho^{\frac{1}{2}}u\|_{L^2} + C$$

$$\le CR_T^{\frac{1}{2}}(\log^{\frac{1}{2}}(e+R_T^{\frac{3}{2}})) + \log^{\frac{1}{2}}(e+\|\nabla u\|_{L^2}^2) + C$$

$A_3$

$$\le CR_T^{\frac{1}{2}}\log(e+A_3^2) + CR_T$$

$$\le CR_T^{\frac{3+k}{2}} \rightarrow C(\alpha,T)R_T^{1+\alpha\beta}$$

$\|\xi\|_{L^\infty}$

$$\log(e+A_3^2) \lesssim C(\alpha,T)R_T^{1+\alpha\beta}$$

$$\|\nabla^{-1}f\|_{L^2} \lesssim \|f\|_{L^q} \quad q<\infty$$

$$q = \frac{2r}{r+1}$$

$\|u\|_{L^2}$

~~$\|\nabla u\|$~~

$$B = |\bar{P} - \bar{F}| \le C + R_T^{\max\{\beta r, 0\}}A_3^2$$

$\rho$ $\quad (2\mu+\lambda)\nabla\cdot u$