§5. Schauder boundary estimate.

Above all, we have done the interior estimate.

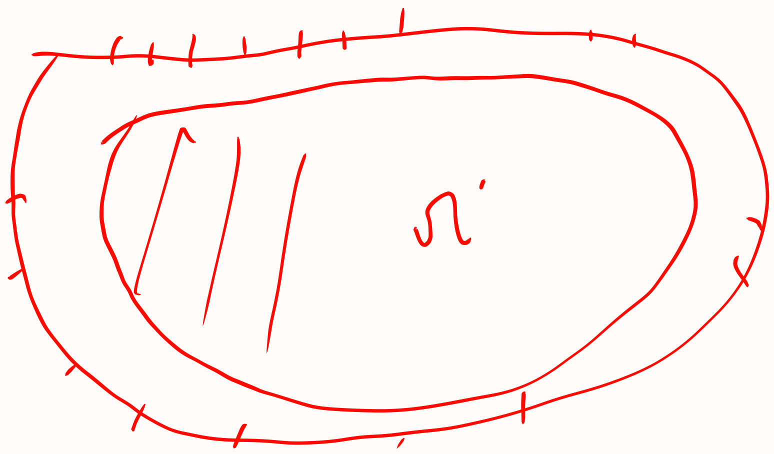
i.e. If $u \in C^{2,\alpha}(\Omega)$ is the solution of $Lu = -a^{ij}D_{ij}u + b^{i}D_{i}u + cu = f$.

$$\begin{cases} ① \ \exists \Lambda \geq \lambda > 0 \text{ s.t. } \lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \ \forall x \in \Omega, \xi \in \mathbb{R}^n \\ ② \ \frac{1}{\lambda}\left\{\sum_{i,j}|a^{ij}|_{\alpha;\Omega} + \sum_i |b^i|_{\alpha;\Omega} + |c|_{\alpha;\Omega}\right\} \leq \Lambda_\alpha \end{cases}$$

$\Omega' \subset\subset \Omega$, we have.

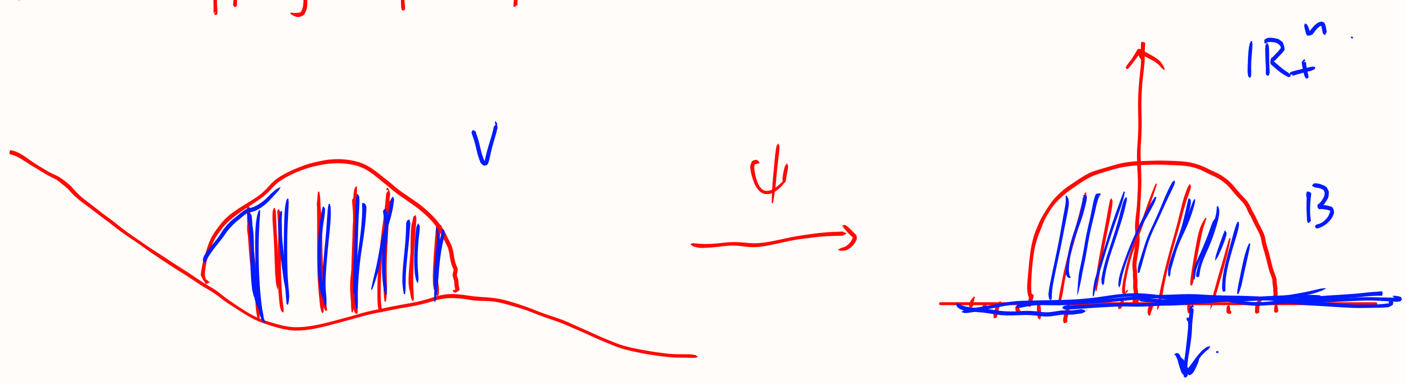
$$\|u\|_{2,\alpha;\Omega'} \leq C(n, \alpha, \frac{\Lambda}{\lambda}, \Lambda_\alpha, \Omega', \Omega)\left[\frac{1}{\lambda}\|f\|_{\alpha;\Omega} + |u|_{0;\Omega}\right].$$

$|u|_{0;\Omega} \nearrow \partial\Omega$



Now we consider the boundary estimate.

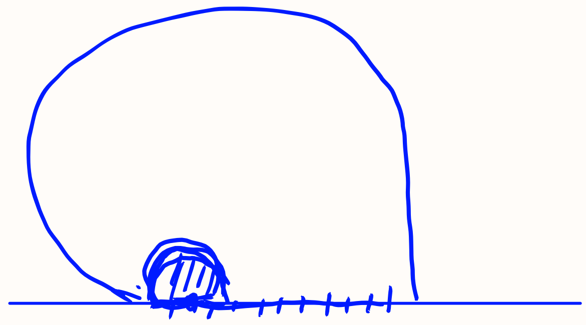
Definition 1. A bounded domain Ω in $\mathbb{R}^n$ and its boundary are of class $C^{k,\alpha}$ $(0 \leq \alpha \leq 1)$. if at each point $x_0 \in \partial\Omega$ there is neighborhood V and one-to-one mapping ψ of V onto $B \subset \mathbb{R}^n$ such that:



(i) $\psi(V \cap \Omega) \subset \mathbb{R}^n_+$. (ii) $\psi(V \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$.

(ii) $\psi \in C^{k,\alpha}(V)$. $\psi^{-1} \in C^{k,\alpha}(B)$

Lemma 5.1 Let Ω be an open subset of $\mathbb{R}^n_+$, with a boundary portion T on $\{x_n = 0\}$.



T

Suppose that $u \in C^{2,\alpha}(\Omega \cup T)$ is a bounded solution in Ω of $Lu = f$

$\boxed{u = 0 \text{ on } T} \Rightarrow$.

$\Rightarrow \exists R_0, C$ (depend on $n, \alpha, \Lambda/\lambda, \Lambda\alpha$).

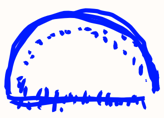
s.t. $\forall 0<R\le R_0$, $B_R^+ \subset \Omega$. whose center is on T.

if $u \in C^{2,\alpha}(\overline{B_R^+})$ and vanishes near $\partial B_R^+ \cap \{x_n > 0\}$.

$C_0^{2,\alpha}(B_R)$

$u \in C_0^{2,\alpha}(\mathbb{R}^n)$.

$-\hat{\Delta u} = \hat{f}$



$-\Delta u = f$.

$-\Delta \hat{u} = -\Delta \hat{u}$.

0

$-\Delta \hat{u} = \hat{f}$.

odd extension.

$\Rightarrow [D^2 u]_{\alpha; B_R^+} \le C \left\{ \frac{1}{\lambda} [f]_{\alpha; B_R^+} + \frac{1}{R^{2+\alpha}} |u|_{0; B_R^+} \right\}$

Lemma 5.2.

Ω is as above $\quad Lu = -a^{ij}D_{ij}u + b^i D_i u + cu = f$

The coefficients satisfy.

① $\exists\ \Lambda \geq \lambda > 0$ s.t. $\quad \lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \ \forall x \in \Omega, \ \xi \in \mathbb{R}^n$.

② $a^{ij}, b^i, c \in C^{0,\alpha}(\bar{\Omega}) \ (0<\alpha<1)$

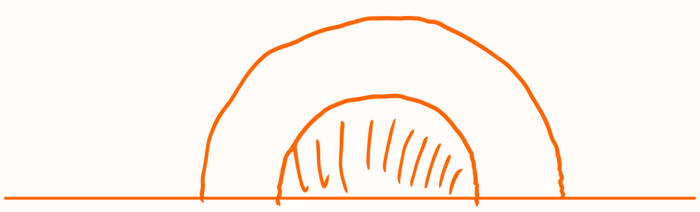
$$\frac{1}{\lambda}\left\{\sum_{ij}|a^{ij}|_{\alpha;\Omega} + \sum_i |b^i|_{\alpha;\Omega} + |c|_{\alpha;\Omega}\right\} \leq \Lambda_\alpha.$$

Let $u \in C^{2,\alpha}(\Omega \cup T)$. $\quad u = 0$ on T.

Then for every $\Omega' \subset\subset \Omega \cup T$.

we have : $\quad |u|_{2,\alpha;\Omega'} \leq C(n, \alpha, \Lambda/\lambda, \Lambda_\alpha, \Omega', \partial\Omega \setminus T)$

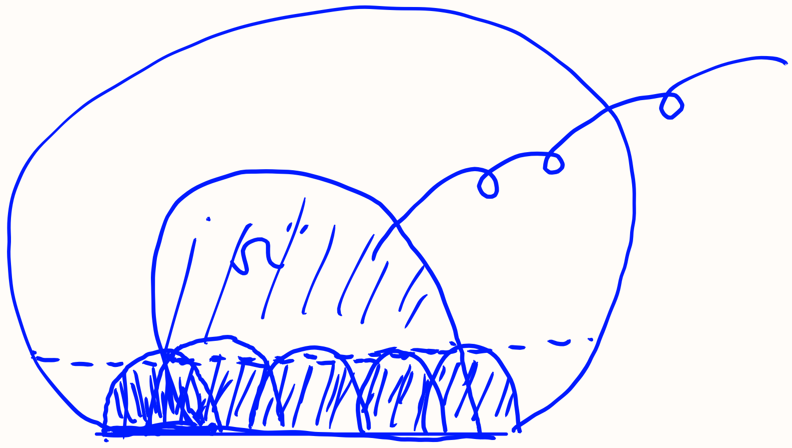
$$\left\{\frac{1}{\lambda}|f|_{\alpha;\Omega} + |u|_{0;\Omega}\right\}.$$



$|u|_{2,\alpha;\frac{B^+}{2}} \leq |f|_{0;B^+} + |u|_{0;B^+}$.

Proof: Without loss of generality. $\lambda = 1$.

$$\begin{cases} \bar{R}_0 = \min\left\{R_0, \frac{1}{2}\operatorname{dist}\{\Omega', \partial\Omega \setminus T\}\right\}, \text{ where } R_0 \text{ is a constant in 3.1} \\ \text{let } \Omega'' = \Omega' \cap \left\{x_n > \frac{1}{4}\bar{R}_0\right\}, \quad \Rightarrow \Omega' \subset\subset \Omega \end{cases}$$

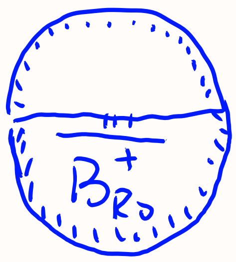


$\Omega' \setminus \Omega''$

Now assume $B_{\bar{R}_0}$ is a ball whose center is in $\bar{\Omega'} \cap T$.

$\overline{B^+_{\frac{\bar{R}_0}{2}}}$ is the covering of $\overline{\Omega'} \setminus \overline{\Omega''}$.

Theorem 4.3.

$\Rightarrow$ 

$u \in C^{2,\alpha}(\overline{B^+_{\bar{R}_0}})$ $\quad \{u\} \in C_0^{2,\alpha}(B^+_{\bar{R}_0})$.

finite covering theorem.

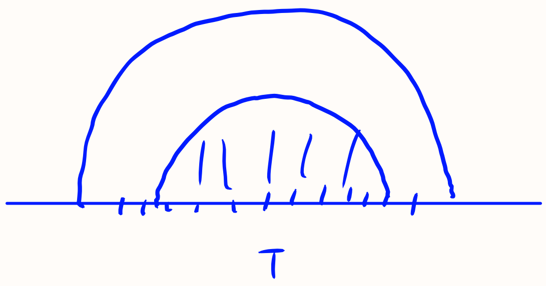
$$\begin{cases} ① \; |u|_{2,\alpha; B^+_{\frac{\bar{R}_0}{2}}} \le C(|f|_{\alpha;\Omega} + |u|_{0;\Omega}). \\ ② \; \Omega'' \text{ interior estimate.} \end{cases}$$

$\overline{\Omega''}$

$$|u|_{2,\alpha;\Omega'} \le C\left\{\frac{1}{\lambda}|f|_{\alpha;\Omega} + |u|_{0;\Omega}\right\}.$$

$$\Omega' = \underline{\Omega'\backslash\Omega''} + \underline{\Omega''}.$$

Corollary I.



$u \in C^{2,\alpha}(B_1 \cup T)$.

$$|u|_{2,\alpha,B^+_{\frac{1}{2}}} \le C\left\{\frac{1}{\lambda}|f|_{\alpha;B_1^+} + |u|_{0;B_1^+}\right\}.$$

Theorem 5.3. (boundary Schauder estimate).

$\partial\Omega \in C^{2,\alpha}$ $(0<\alpha<1)$. $\qquad c \ge 0$

$u \in C^{2,\alpha}(\bar\Omega)$. satisfy $Lu = -a^{ij}D_{ij}u + b^i D_i u + cu = f$.

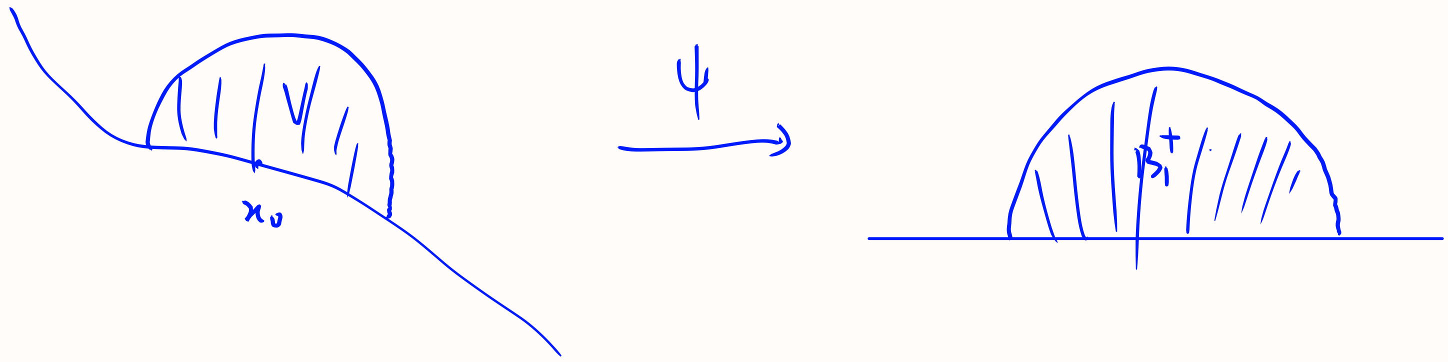
$u = 0$ on $\partial\Omega$.

$$\Rightarrow |u|_{2,\alpha;\Omega} \le C(n,\alpha,\frac{\Lambda}{\lambda},\Lambda_\alpha,\Omega)\left[\frac{1}{\lambda}\|f\|'_{\alpha;\Omega} + \|u\|_{0;\Omega}\right]$$

Proof: Let $\lambda = 1$. Assume $x_0 \in \partial\Omega$.

ψ is the heomomorplism of $x_0 \in \partial\Omega$.

$\Rightarrow |\psi|_{2,\alpha,V\cap\Omega} \le K$, $|\psi^{-1}|_{2,\alpha;B_1^+} \le K$



$y = \psi(x)$

$$\begin{cases} \bar{u}(y) = \bar{u}(\psi(x)) = u(x) \\ u(x) = u(\psi^{-1}(y)) = \bar{u}(y) \end{cases}$$

$$-a^{ij} D_{ij} u + b^i D_i u + cu = f$$

$$D_i u(x) = \sum_k \bar{u}_k \frac{\partial y_k}{\partial x_i}, \quad D_{ij} u(x) = \sum_{k,m} \bar{u}_{km} \frac{\partial y_m}{\partial x_j} \frac{\partial y_k}{\partial x_i} + \sum_k \bar{u}_k \frac{\partial^2 y_k}{\partial x_i \partial x_j}$$

$\downarrow$

$$\bar{a}^{rs} D_{rs} \bar{u} + \bar{b}^r D_r \bar{u} + \bar{c}\bar{u} = \bar{f}$$

$$\begin{cases} \bar{a}^{rs} = -a^{ij} \frac{\partial y_r}{\partial x_i} \frac{\partial y_s}{\partial x_j} \\ \bar{b}^r = -a^{ij}(x) \frac{\partial^2 y_r}{\partial x_i \partial x_j} + b^i \frac{\partial y_r}{\partial x_i} \\ \bar{c}(y) = c(x) \end{cases}$$

$$A = \left(\frac{\partial y_r}{\partial x_i}\right)_{ri}$$

① uniformly elliptic ✓ $|\psi', \psi^{-1}| \le K$.

$\psi \in C^{2,\alpha}$

② $\frac{1}{\lambda}\left[\sum |\bar{a}^{rs}|_{\alpha;B_1} + \sum |\bar{b}^r|_{\alpha;B_1} + |\bar{c}|_{\alpha;B_1}\right] \le \Lambda'_\alpha$ ✓

$$(\bar{a}^{rs})_{rs} = -A^T (a^{ij})_{ij} A$$

$\downarrow$

λ, Λ.

$$-\lambda |A\xi|^2 \geq \bar{a}^{rs}\xi_r\xi_s = \xi^T(\bar{a}^{rs})_{rs}\cdot\xi \qquad (A\xi)_i$$

$$= -\underset{\sim}{(A\xi)}^T(a^{ij})_{ij}\underset{\sim}{(A\xi)}$$

$$\geq -\Lambda |\underset{\sim}{A}\xi|^2 \qquad \underset{\sim}{A^T}A$$

$$0 < s_1 \leq |A| \leq s_2$$

$$\frac{1}{k} \leq |A| \leq k$$

$$\underline{C^{0,\alpha}} \qquad \underset{\sim}{\varphi \in C^{2,\alpha}} \qquad [ab]_\alpha \leq [a]_\alpha|b|_0 + [b]_\alpha|a|_0 \checkmark$$

② $\bar{b}^r = \dfrac{\partial^2 y_r}{\partial x_i \partial x_j}$ $\quad [\bar{a}^{rs}]_\alpha$

$[\bar{b}^r]_\alpha \leq$ ✓

$$|\bar{u}|_{2,\alpha;B^+_{\frac12}} \leq C(|\bar{u}|_{0;B_1^+} + [\bar{f}|_{\alpha;B_1^+}).$$

$$\underset{\sim}{|u|_{2,\alpha;\psi^{-1}(B^+_{\frac12})} \leq C(|u|_{0;\Omega} + |f|_{\alpha;\Omega}).} \quad \checkmark$$

Remark 1. If $u=\varphi$ on $\partial\Omega$ $\quad \varphi\in C^{2,\alpha}(\bar{\Omega})$. $\quad \underline{C^{2,\alpha}}$

$$\Rightarrow \underline{\|u\|_{2,\alpha,\Omega} \leq c\left[\boxed{\|f\|_{\alpha;\Omega}} + \boxed{|\varphi|_{2,\alpha;\Omega}} + |u|_{0;\Omega}\right]}. \quad \checkmark$$

$w = u - \varphi$ $\qquad \underline{f}\cdot\underline{\varphi}$.

Remark 2. let $u\in C^{2,\alpha}(\bar{\Omega})$. $\underline{a^{ij}, b^i, c, f\in C^{k,\alpha}(\bar{\Omega})}$. $\underline{\partial\Omega\in C^{k+2,\alpha}}$. $\underline{\varphi\in C^{k+2,\alpha}}(\bar{\Omega})$

$\Rightarrow u \in C^{k+2,\alpha}(\bar{\Omega})$

$$\|u\|_{C^{k+2,\alpha}(\bar\Omega)} \le C(\|f\|_{k,\alpha} + \|\varphi\|_{k+2,\alpha} + |u|_{0;\Omega}).$$

§6. Maximum Principle

Theorem 1. (weak maximum principle) $Lu := -a^{ij}D_{ij}u + b^i D_i u + cu$.

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2. \qquad 0<\lambda\le\Lambda.$$

b^i is bounded, $c \ge 0$. if $u \in C^2(\Omega)\cap C(\bar\Omega)$. and satisfy

$$Lu \le f \quad \text{in } \Omega.$$

$$\Rightarrow \sup_\Omega u \le \sup_{\partial\Omega} u^+ + C(n, \tfrac{1}{\lambda}\textstyle\sum |b^i|_{0;\Omega}, \operatorname{diam}\Omega)\, |f|_{0;\Omega}.$$

proof:

step 1.

(i) $c(x) \ge c_0 > 0$. let $v = u - \sup_{\partial\Omega} u^+$ $\Rightarrow$

v satifies: $\begin{cases} Lv \le f - c\sup_{\partial\Omega} u^+ \le f. & \text{in } \Omega \\ v \le 0 & \text{on } \partial\Omega \end{cases}$

If v get the nonnegative maximal in Ω (x_0)

$\begin{cases} D^2 v(x_0) \text{ negative definite.} & Dv(x_0) = 0. \\ \text{trace } D^2 v(x_0) \le 0. \end{cases}$

$$\Rightarrow \left[\underbrace{-a^{ij} D_{ij} v}_{\parallel} + \underline{b^i D_i v} \right]_{x_0} \geq 0.$$

$$\text{trace}\left(\underset{\downarrow}{(a_{ij})}\, \underset{\downarrow}{D^2 v} \right).$$

$$\underline{c(x_0)} v(x_0) \leq Lv \leq |f|_{0;\Omega}.$$

$$\Rightarrow \sup_{\Omega} v \leq \frac{|f|_{0,\Omega}}{c_0}.$$

$$\Rightarrow \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \frac{1}{c_0} |f|_{0;\Omega}.$$

(ii) $c(x) \geq 0$.

$v = w \cdot z$ $\qquad$ $z > 0$ is pending. $\qquad$ $\underline{w|_{\partial\Omega} \leq 0}$.

$Lv \to \underline{Lw}$

$$-a^{ij} D_{ij} w + \left(b^i - \frac{2}{z} a^{ij} \frac{\partial z}{\partial x_j} \right) D_i w + \underbrace{\left[c + \frac{1}{z}\left(b^i D_i z - a^{ij} D_{ij} z \right) \right]}_{\geq c_0 > 0,} w \leq \frac{f}{z}.$$

$$\underline{\frac{1}{z}\left(b^i D_i z - a^{ij} D_{ij} z \right) \geq 0}.$$

$$z: \begin{cases} z > 0 \\ (b^i D_i z - a^{ij} D_{ij} z) > 0 \end{cases}$$

$$z = e^{2\alpha d} - e^{\alpha x_1}$$

Without loss generality. $\underline{\Omega \subset \{ x : 0 < x_1 < d \}}$.

$-a^{ij}D_{ij}z + b^iD_iz$.

$= -a^{11}D_{11}z + b^1D_1z = a^{11}\alpha^2 e^{\alpha x_1} - \underline{b^1}\alpha e^{\alpha x_1} \underline{> 0}$ (α is sufficiently large)

$w\big|_{\partial\Omega} = 0$. $\Rightarrow \sup_{\Omega} w \le C|f|_{0;\Omega}$.

$\Rightarrow$ Theorem 6.2. $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

$$\underline{|u|_{0;\Omega} \le \sup_{\partial\Omega}|u| + C|f|_{0;\Omega}}$$

$\qquad\qquad \| \qquad\qquad \|$

$\qquad \|\varphi\|_0 \qquad \|f\|_0$.

$\underline{u}$ and $\underline{-u}$.

$\underline{C \geq 0}$.