Global regularity of Vaigant-Kazhikhov model 6.

$$D_t\theta + D_t\xi + p + m + B = 0.\qquad \begin{cases} \theta = \theta(\rho),\ \theta' = \frac{2\mu+\lambda}{\rho}, \\ \xi = (\Delta^{-1})\nabla\cdot(\rho u), \\ m = [u^i, R_iR_j](\rho u^j), \\ B = \bar{p} - \bar{F}, \qquad , \qquad \mathbb{T}^2, \end{cases}$$

$$F = (2\mu+\lambda)\nabla\cdot u - (p - \bar{p})$$

estimate of $R_T = \|\rho\|_{L^\infty_T L^\infty}$

$\longrightarrow$

uniform bound: $R_T \le C$.

$\beta > \frac{3}{2},\ \gamma \in (1, 4\beta-3]$,

optimal bound: $R_T \le C(T)$,

$\beta > \frac{4}{3},\ \gamma > 1$.

$\tilde{C}$

$k = \max\{0,\ \gamma-2\beta,\ \beta-\gamma-2\}$.

$\|\rho\|_{L^\infty_T L^q} \le C(T)$.

	uniform bound	optimal bound
m	$\|m\|_{L^\infty} \le C(\varepsilon)\left(\frac{R_T^{-1-k}A_2^2}{e+A_1^2} + R_T^{\frac{3+k}{2}+\varepsilon}A_3^{\frac{3}{2}} + R_T^{1+\varepsilon}\right)$	$\|m\|_{L^\infty} \le C(\varepsilon, T) R_T^{1+\frac{\beta}{4}+\varepsilon}(1+A_3^2) + \frac{A_2^2}{e+A_1^2}$.
ξ	$\|\xi\|_{L^\infty} \le C(\alpha) R_T^{\frac{2+k+\alpha\beta}{2}}$	$\|\xi\|_{L^\infty} \le C(\alpha, T) R_T^{\frac{2+\alpha\beta}{2}}$
B	$\|B\| \le C(1 + R_T^{\max\{\beta-\gamma, 0\}} A_3^2)$	$\|B\| \le C(T)(1 + A_3^2)$.
A_i	$\sup \log(e + A_1^2 + A_3^3) + \int_0^T \frac{A_2^2}{e+A_1^2} \le C(\alpha) R_T^{1+k+\alpha\beta}$	$\sup \log(e + A_1^2 + A_3^3) + \int_0^T \frac{A_2^2}{e+A_1^2} \le C(\alpha, T) R_T^{1+\alpha\beta}$

uniform

$$\|m\|_{L^\infty} \lesssim \|\nabla u\|_{L^2} \oplus \|\nabla u\|_{L^q} \oplus \|\rho u\|_{L^q}$$

$$\|\rho^{\frac{1}{2}} u\|_{L^2} \oplus \underbrace{\|u\|_{L^\infty}} \lesssim (\|\nabla u\|_{L^2} \oplus \log(e + \|\nabla u\|))$$

$$\|\rho u\|_{L^{2+}} \oplus \underbrace{\|\rho u\|_{L^r}} \lesssim (\|\rho u^2\|_{L^1} \oplus \|\nabla u\|_{L^2})$$

γ

Zlotnik inequality:

$\theta(t)$

$y(t)$, $y(0) = y_0$,

$y'(t) = \underbrace{g(y)(t)} + \underbrace{b'(t)}$.

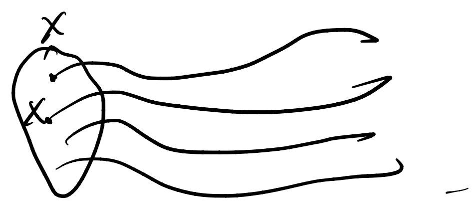
① $g(+\infty) = -\infty$, $g \in C(\mathbb{R})$,

② $b(t_2) - b(t_1) \le N_0 + N_1 (t_2 - t_1)$,

$\Rightarrow$ $y(t) \le \bar{y} =: \max\{y_0, \hat{\zeta}\} + N_0$, $\hat{\zeta} = \hat{\zeta}(N_1)$, $g(\zeta) \le -N_1$, $\forall \zeta \ge \hat{\zeta}$.

$\underline{D_t \theta} = -p - D_t \xi - m - B$.

Lagrange coordinate: $f(x,t) = \hat{f}(\underline{x}, t)$,



$\partial_t \hat{\theta} = -\underline{\hat{p}} - \partial_t \hat{\xi} - \hat{m} - B$. $\quad p = p(\rho) = p(\theta)$. $\quad \theta \sim \rho$, $(\theta' = \frac{2\mu+\lambda}{\rho})$.

$= -\hat{p}(\theta) - \partial_t \underbrace{(\hat{\xi} + \int_0^t \hat{m} + \int_0^t B)}$ $\longrightarrow b$.

uniform estimate:

$|b(t_2) - b(t_1)| \le \underbrace{2\|\xi\|_{L^\infty}} + \int_{t_1}^{t_2} \|m\|_{L^\infty} + \int_{t_1}^{t_2} |B|$.

$\le C(\alpha) R_T^{\frac{2+k+\alpha\beta}{2}} + C(\alpha)\left(R_T^{\alpha\beta} + R_T^{\frac{3+k}{2}+\varepsilon} + R_T^{1+\varepsilon}(t_2 - t_1)\right)$

$C(\alpha, T) R_T^{\frac{2+\alpha\beta}{2}}$

$+ C R_T^{\max\{\beta-\gamma, 0\}} + C(t_2 - t_1)$. $\quad C(\alpha,T)\left(R_T^{1+\alpha\beta} + R_T^{1+\frac{\beta}{\varphi}+\varepsilon}\right)$.

$C(\alpha,T)(1 + (t_2 - t_1))$

$\le \underbrace{C(\alpha) R_T^{\max\{\frac{3+k}{2}+\varepsilon, \beta-\gamma\}}} + \underbrace{C_{(\varepsilon)} R_T^{1+\varepsilon}}_{N_1} (t_2 - t_1)$.

$N_0 = C(\alpha,T) R_T^{1+\frac{\beta}{\varphi}+\varepsilon} = \underline{N_1}$

$p \qquad \rho \qquad \theta \qquad \hat{\zeta} = \underline{R_T^{\frac{\beta}{\gamma}(1+\frac{\beta}{\varphi}+\varepsilon)}} \longrightarrow$

$-p \le -C(\varepsilon) R_T^{1+\varepsilon} \Leftrightarrow \rho \ge C(\varepsilon) R_T^{\frac{1+\varepsilon}{\gamma}} \Leftrightarrow \theta \ge \underline{C(\varepsilon) R_T^{\frac{\beta(1+\varepsilon)}{\gamma}}}$.

$\uparrow$

$\hat{\zeta}$

R_T^{β}

$$\le \rho^{\beta} \le \hat{\theta} \le \max\{C(\rho_0), C(\varepsilon) R_T^{\frac{\beta(1+\varepsilon)}{\gamma}}\} + C(\varepsilon) R_T^{\max\{\frac{3+k}{2}+\varepsilon, \beta-\gamma\}}$$

sup sup

$$\le R_T^{\max\{\frac{3+k}{2}+\varepsilon, \beta-\gamma, \frac{\beta(1+\varepsilon)}{\gamma}\}}$$

R_T

$R_T^{\max\{1+\frac{\beta}{\varphi}+\varepsilon, \frac{\beta}{\gamma}(1+\frac{\delta}{\varphi}+\varepsilon)\}}$.

$$\begin{cases} \beta > \frac{3+k}{2}+\varepsilon, \Longleftarrow \begin{cases} \beta > \frac{3}{2}+\varepsilon & \Longleftarrow \beta > \frac{3}{2}+\varepsilon. \\ \beta > \frac{3+\gamma-2\beta}{2}+\varepsilon. & \Longleftarrow \gamma < 4\beta-3-2\varepsilon. \\ \beta > \frac{3+\beta-\gamma-2}{2}+\varepsilon & \Longleftarrow \checkmark \quad \varepsilon - \end{cases} \\ \beta > \beta-\gamma, \checkmark \\ \beta > \frac{\beta(1+\varepsilon)}{\gamma} \Longleftarrow \gamma > 1+\varepsilon \end{cases}$$

$\varepsilon \to 0$

$\gamma > 1$.

$\beta > \frac{3}{2}, \ \gamma < 4\beta-3-2\varepsilon.$

$$\begin{cases} \beta > 1+\frac{\beta}{\varphi}+\varepsilon, \Longrightarrow \beta > \frac{\varphi}{3}+C\varepsilon \\ \beta > \frac{\beta}{\gamma}(1+\frac{\beta}{\varphi}+\varepsilon) \end{cases}$$

local existence $\longrightarrow$ $\|\rho\|_{L^\infty}$ bound $\xrightarrow{2D}$ $\|\rho\|_{H^1\cap W^{1,q}}, \|u\|_{H^2} \le C.$ $\longrightarrow$ $\|\rho\|_{H^3}, \|u\|_{H^3} \le C$

contraction mapping ✓ strong classic.

$\downarrow$

Galerkin method

$\downarrow$

blow up criterion

Huang-Li (2011). Serrin blowup criterion: 3D

(ρ, u) blowup at T^*,

$$\lim_{T\to T^*} \|\rho\|_{L_T^\infty L^\infty} + \|u\|_{L^s L^q} = +\infty.$$

if $\mu > 7\lambda$,

$$\lim_{T\to T^*} \|\rho\|_{L_T^\infty L^\infty} = +\infty.$$

$|\nabla\rho|^q$ equation: $|\nabla\rho|^q|\nabla u|$. $\rho|\nabla\rho|^{q-1}|\nabla^2 u|$

$$\partial_t|\nabla\rho|^q + \underline{\nabla\cdot(|\nabla\rho|^q u) + (q-1)|\nabla\rho|^q\nabla\cdot u + q|\nabla\rho|^{q-2}\nabla\rho\cdot\nabla u\cdot\nabla\rho} + q\rho|\nabla\rho|^{q-2}\nabla\rho\cdot\nabla(\nabla\cdot u) = 0$$

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \leq C\|\nabla\rho\|_{L^q}\|\nabla u\|_{L^\infty}^{\frac{1}{q}} + C\|\nabla\rho\|_{L^q}^{\frac{q-1}{q}}\|\nabla^2 u\|_{L^q}^{\frac{1}{q}}$$

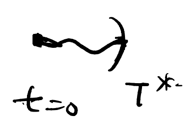
$$\leq C\|\nabla\rho\|_{L^q}(\|\nabla u\|_{L^\infty}+1) + C(\|\nabla\dot u\|_{L^2}+1).$$

Beale-Majda-Kato: (1964). blowup of incompressible Navier-Stokes

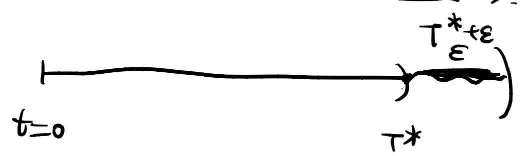
$$\|\nabla u\|_{L^\infty} \leq C(\|\nabla\cdot u\|_{L^2} + \|\nabla\times u\|_{L^2})\log(e+\|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C.$$

$\rho_0 \in W^{1,q}$, $u_0 \in H^2$, $\rho\dot{u}_0 \in L^2$

$\downarrow$



$\|\rho\|_{W^{1,q}}$, $\|u\|_{H^2} \leq C$



$\boxed{\rho(T^*) \in W^{1,q}, \quad u(T^*) \in H^2}$