The Hahn-Banach Theorem

§1.1. 分析形式

Def1. (次线性泛函). $p: E \to \mathbb{R}$ 满足

(i) $p(\lambda x) = \lambda p(x)$, $\forall x \in E$, $\boxed{\lambda > 0}$

(ii) $p(x+y) \le p(x) + p(y)$, $\forall x, y \in E$.

Thm1 E 是实向量空间, $F \subset E$ 子空间, 余维为1. $p: E \to \mathbb{R}$ 次线性泛函.

f 是 F 上的线性泛函, f 满足 $f(x) \le p(x)$, $\forall x \in F$.

则 $\exists\ \tilde{f}: E \to \mathbb{R}$ 线性泛函, s.t. $\tilde{f}|_F = f$, $\boxed{\tilde{f}(x) \le p(x)}$ $\forall x \in E$.

$|\tilde{f}(x)| \le |p(x)|$

Pf. 任取 $x_0 \in E \setminus F$, $E = F + \mathbb{R}\{x_0\}$

定义 $\tilde{f}(tx_0 + x) = t\tilde{f}(x_0) + \tilde{f}(x) = ta + f(x)$, $\forall t \in \mathbb{R}, x \in F$.

$\tilde{f} \le p \iff ta + f(x) \le p(tx_0 + x)$, 即 $ta \le p(tx_0 + x) - f(x)$

$\forall t \in \mathbb{R}, x \in F$ 成立

当 $t > 0$ $\quad a \le \frac{1}{t} p(tx_0 + x) - \frac{1}{t} f(x)$

$t < 0$ $\quad a \ge \frac{1}{t} p(tx_0 + x) - \frac{1}{t} f(x)$

$$\sup_{\substack{s<0\\ y \in F}} \frac{1}{s} p(sx_0 + y) - \frac{1}{s} f(y) \le \boxed{a} \le \inf_{\substack{t>0\\ x \in F}} \frac{1}{t} p(tx_0 + x) - \frac{1}{t} f(x)$$

$\forall s < 0, t > 0, x, y \in F$. $\quad \frac{1}{s} p(sx_0 + y) - \frac{1}{s} f(y) \le \frac{1}{t} p(tx_0 + x) - \frac{1}{t} f(x)$

$\iff \frac{1}{t} p(tx_0 + x) - \frac{1}{s} p(sx_0 + y) \ge -\frac{1}{s} f(y) + \frac{1}{t} f(x)$

$\Longleftarrow p(x_0 + \frac{x}{t} - x_0 - \frac{y}{s}) = p(\frac{x}{t} - \frac{y}{s}) \ge f(\frac{x}{t} - \frac{y}{s})$

$\frac{x}{t} - \frac{y}{s} \in F$. #

Thm2 (H-B 定理) E, $F \subset E$ 子空间, f 是 F 上的线性泛函, p 如上.

$f: F \to \mathbb{R}$ 线性, $f \le p$.

则存在 $\tilde{f}: E \to \mathbb{R}$ 线性, 且 $\tilde{f} \le p$.

Pf. 考虑 (G, g) 二元组构成. (G, g) 满足

(1). $F \subset G \subset E$, G是E的子空间

(2). g 满足. g是G上的线性泛函, $g|_F = f$.

定义 $\mathcal{F}$ 上一偏序关系 $\leq$.

$$(G, g) \leq (H, h) \iff G \subset H, \quad h|_G = g.$$

则$\mathcal{F}$的任一全序子集有上界. $\mathcal{G} \subset \mathcal{F}$. 全序.

令 $H = \bigcup_{G \in \mathcal{G}} G$. $\quad h: H \to \mathbb{R}$, 若 $x \in G$. 定义 $h(x) = g(x)$

$(G, g) \leq (H, h)$ 由Zorn引理, $\mathcal{F}$有极大元 (M, m)

则 $M = E$. 若不然. $\exists\, x_0 \in E \setminus M$. 考虑 $M \cup \mathbb{R}\{x_0\}$, $m \to \tilde{m}$

$(M \cup \mathbb{R}\{x_0\}, \tilde{m}) \geq (M, m)$ 矛盾. #

Thm. (H-B 复). E, F 同上. 复. $p: E \to \mathbb{R}$. 半范. $\boxed{f}: F \to \mathbb{K}$. 线性 且 $|f| \leq p$

则存在 $\tilde{f}: E \to \mathbb{K}$ 线性. $|\tilde{f}| \leq p$.

Remark. E上的复线性泛函和实线性泛函间的

$$f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x) \qquad \operatorname{Re} f(x) \text{ 记为 } \varphi(x)$$

$$f(ix) = i\operatorname{Re} f(x) - \operatorname{Im} f(x).$$

则 $\operatorname{Im} f(x) = \varphi(ix) \Rightarrow \operatorname{Im} f(x) = -\varphi(ix)$

$$f = \varphi(x) - i\varphi(ix)$$

f. 复 $\quad \varphi = \operatorname{Re} f$. E 实 $\quad p$. $\quad \varphi \to \tilde{\varphi}$.

$$\tilde{f} = \tilde{\varphi}(x) - i\tilde{\varphi}(ix)$$

(赋范)

Cor 1. E, F 同上. $g: F \to \mathbb{R}$ 的连续线性泛函. 则 $\exists\, f: E \to \mathbb{R}$ 连续线性泛函,

且 $\|f\|_{E^*} = \|g\|_{F^*}$.

Pf: 考虑 F. $|g(x)| \leq \|g\|_{F^*}\|x\|$ $\qquad \|g\|_{F^*}\|\cdot\|$

由H-B定理. $\exists\, f \in E^*$ s.t.

$$|f(x)| \leq \|g\|_{F^*}\|x\| \quad \Rightarrow \quad f \in E^*. \text{ 且 } \|f\|_{E^*} \leq \|g\|_{F^*}$$

又 $f|_F = g$. $\quad \|f\|_{E^*} \geq \|g\|_{F^*} \quad \Rightarrow \quad \|f\|_{E^*} = \|g\|_{F^*}$

Cor 2. E 同上, $\forall x \in E$, $x \neq 0$. $\exists f \in E^*$. s.t. $f(x) = \|x\|$, 且 $\|f\| = 1$

pf: 考虑 $\mathbb{R}\{x\}$. 定义 $g(tx) = t\|x\|$. g 线性, 连续. $\|g\| \leq 1$. $\Rightarrow \|g\| = 1$.

由 Cor 1. $\exists f \in E^*$, s.t. $\|f\|_{E^*} = 1$, 且 $f(x) = g(x) = \|x\|$. 丹

eg. $\sigma(E, E^*)$ 是 Hausdorff 的.

$E^* = \{f_i\}_{i \in I}$. $\{|f_i|\}_{i \in I}$ 是 E 上一族半范, 定义"开球"

选取 $J \subset I$. 有限子集,

$$B_J(x, \delta) = \{y \in E \mid \max_{i \in J} |f_i(x-y)| < \delta\}.$$

开球作为基生成的拓扑 即为 $\sigma(E, E^*)$

由于 Cor 1, $\forall x \neq y$. $\exists f \in E^*$. s.t. $f(x) \neq f(y)$ $\Rightarrow$ $\sigma(E, E^*)$ 是 Hausdorff.

eg. $\underset{\sim}{\iota}: E \to E^{**}$. $x \longmapsto x^{**}$. $\|x\|_E = \|x^{**}\|_{E^{**}}$

$$\|x^{**}\|_{E^{**}} = \sup_{\|f\|_{E^*} \leq 1} \langle x^{**}, f \rangle$$

$$= \sup_{\|f\|_{E^*} \leq 1} \langle f, x \rangle$$

$$= \|x\|_E$$

$$\boxed{\|x\| = \sup_{\|f\|_{E^*} \leq 1} |\langle f, x \rangle|}$$

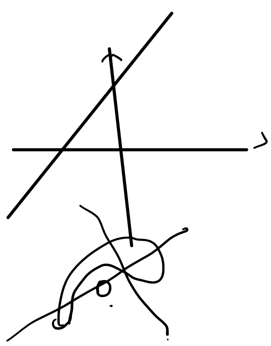
$X \to Y$ $\mathcal{B}(X, Y)$ $B(X, Y)$ 完备 $\Rightarrow$ Y 完备

§1.2 几何形式之 H-B 定理. 凸集隔离定理

Def 1. E 上之一个超平面定义为 $H = \{x \in E : f(x) = \alpha\}$.

eg. $\mathbb{R}^2$. $f(a\vec{e_1} + b\vec{e_2}) = af(\vec{e_1}) + bf(\vec{e_2}) = \alpha$.

$f(x) = \alpha$. $\Leftrightarrow$ (a, b). 确定



prop 1. H : $[f = \alpha]$ 闭 $\Leftrightarrow$ f 连续

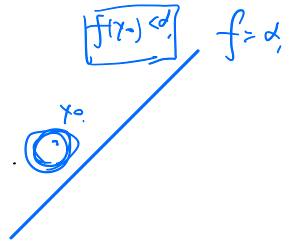
pf: "$\Leftarrow$" 显然

"$\Rightarrow$" 不妨假设 $\alpha = 0$ $[f = 0]$ 闭 $\Rightarrow$ $[f = \alpha]$ 闭

$\forall x \in E$, $f(x) \neq \alpha$. $\forall f(x_0) = \alpha$. $f(x - x_0) \neq 0$

$\exists B(x-x_0, \delta)$. s.t. $f(B(x-x_0,\delta)) \neq 0$, 则

$f(B(x,\delta)) \neq f(x_0) = \alpha$



$\forall\, x_0 \in H^c$, $f(x_0) \neq \alpha$. 不妨假设 $f(x_0) < \alpha$

$B(x_0, r) \subset H^c$.

$f(x) < \alpha$, $\forall\, x \in B(x_0, r)$

反之，若 $\exists\, x_1 \in B(x_0, r)$. $f(x_1) > \alpha$, $\exists\, \bar{x} \in \overline{x_0 x_1}$, s.t. $f(\bar{x}) = \alpha$. 矛盾.

$f(x_0 + r z) < \alpha \quad \forall z \in B(0,1)$

$\|f\| \leq \frac{1}{r}(\alpha - f(x_0))$

反证. 假设 f 不连续, $\exists\, x_n$. s.t. $|f(x_n)| \geq n\|x_n\|$

$y_n = \frac{x_n}{f(x_n)}$, 则 $f(y_n) = 1$, $\Rightarrow \|y_n\| \leq \frac{1}{n}$. $y_n \to 0$.

又由 $[f=1]$ 闭. $\Rightarrow f(0) = 1$. 矛盾. #

Def 2. $A, B \subset E$.

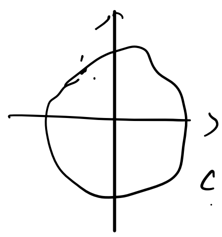
称 $[f=\alpha]$ 分离 A, B. 若 $f(x) \leq \alpha$, $\forall x \in A$, $f(y) \geq \alpha$, $\forall y \in B$.

严格分离: $\exists \varepsilon$. $f(x) \leq \alpha - \varepsilon$ $\forall x \in A$, $f(y) \geq \alpha + \varepsilon$, $\forall y \in B$.

Thm 1 (凸集隔离定理) $A, B \subset E$, 凸. A开, 则存在超平面分离 A, B.

Def 3. (Minkowski functional) $C \subset E$, 包含0的开凸集.

$$p(x) = \inf\{\alpha > 0 : \frac{x}{\alpha} \in C\}$$



(1) $C = \{x \in E : p(x) < 1\}$

(2) $\exists M$. s.t. $0 \leq \boxed{p(x) \leq M\|x\|}$

$\boxed{\exists B(0,r) \subset C}$ $\quad p(x) \leq \frac{1}{r}\|x\|$, $\forall x \in E$.

$$p(x) \leq \frac{1}{r}\|x\| \Leftarrow \frac{x}{\frac{1}{r}\|x\|} \in C \Leftarrow r\frac{x}{\|x\|} \in C$$

(3) $p(x)$ 是次线性泛函.

Lem 1. $C \subset E$, 非空开凸集, 包含0. $x_0 \notin C$, $\exists f \in E^*$. s.t. $f(x) < f(x_0)$, $\forall x \in C$

Pf. $\mathbb{R}\{x_0\}$ 定义 $g(tx_0) = t \le p(tx_0)$. $\forall x_0 \notin C$

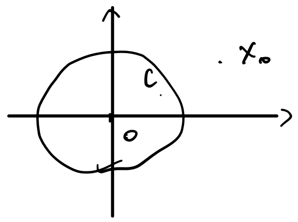
$$g(x_0) = 1 \le p(x_0) \Rightarrow g(tx_0) \le p(tx_0) \le M\|tx_0\|$$

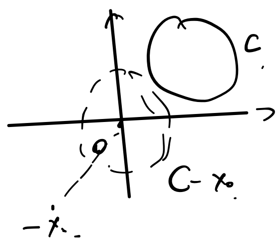
由 H-B Thm, $\exists f \in E^*$ s.t. $f(x_0) = g(x_0) = 1$.

$$f(x) \le p(x) < 1, \quad x \in C$$

f 即为所求. #

特别地, $0 \notin C$, $\exists f \in E^*$ s.t. $f(x) < f(0) = 0$, $\forall x \in C$





$\forall x_0 \in C$, $C \to C - x_0$

$0 \to -x$

$\exists f \in E^*$.

s.t. $f(C - x_0) < f(-x_0)$

$f(C) < 0$

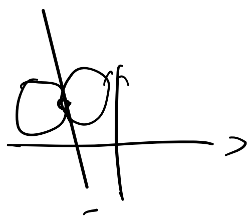
$$\boxed{A-B := \{a-b \mid a \in A, b \in B\}} = \bigcup_{b \in B} (A-b)$$

proof of Thm 1. 令 $C = A - B$, 凸, 开, 由于 $A \cap B = \emptyset \Rightarrow 0 \notin C$.

$\exists f \in E^*$. s.t. $f(z) < 0$, $\forall z \in C$.

$f(x-y) < 0$, $\forall x \in A, y \in B$.

$f(x) < f(y)$, $x \in A, y \in B$.



$\exists$ 超平面 $[f = \alpha]$ 分离 A, B.

Thm 2. (严格隔离定理). $A, B \subset E$, $A \cap B = \emptyset$. A 闭, B 紧, 则存在超平面严格分离 A, B

Pf: $A - B$ 凸, 闭 ($\forall a_n - b_n \to x$, $b_{n_k} \to b$, $a_{n_k} \to x + b$, $x + b \in A$, $x \in A - B$)

$0 \notin A - B$. 存在 $B(0,r)$. s.t. $B(0,r) \cap (A-B) = \emptyset$

则存在 $f \in E^*$. $[f = \alpha]$ 分离 $B(0,r)$ 与 $A - B$.

$$f(a-b) \le f(rz), \quad \forall z \in B(0,1), \ a \in A, b \in B$$

$$f(a-b) \le -r\|f\|, \quad 令 \ \varepsilon = \tfrac{1}{2} r\|f\|. \quad f(x) + \varepsilon \le f(y) - \varepsilon.$$

Remark. 无限维时, A, B 凸. $A \cap B \neq \emptyset$ 可分离

有限维 ?

Cor 2. $\mathcal{X}$ 赋范, $F\subset\mathcal{X}$ 子空间. $\overline{F}\neq\mathcal{X}$, 则存在 $f\in\mathcal{X}^*$, $f\neq 0$, 且 $f|_F=0$.

Remark: $\forall f\in\mathcal{X}^*$ $f|_F=0 \Rightarrow f\equiv 0$. 则 $\overline{F}=\mathcal{X}$.

pf. $\forall x_0\notin\overline{F}$. $\{x_0\}$ 紧, $\overline{F}$ 闭, $\exists f\in\mathcal{X}^*$ s.t. $f(x_0)-\varepsilon > f(x)+\varepsilon$, $\forall x\in\overline{F}$.

$f(x) < \boxed{f(x_0)}$, $\forall x\in\overline{F}$, 又 F 子空间 $\Rightarrow f|_{\overline{F}}=0$

($\exists y\in\overline{F}$, $f(y)\neq 0$: $f(\alpha y)\to\infty$ 矛盾)

eg. $\overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{-k,p'}(\Omega)}} = W^{-k,p'}(\Omega)$

其中 $W^{-k,p'}(\Omega) := \left(W_0^{k,p}(\Omega)\right)^*$, $1<p<\infty$,

$$\|u\|_{W^{-k,p'}(\Omega)} := \sup_{\|v\|_{W^{k,p}(\Omega)}\le 1} |\langle u,v\rangle|.$$

1. $C_c^\infty(\Omega)\subset W^{-k,p'}(\Omega) = \left(W_0^{k,p}(\Omega)\right)^*$.

fix $\varphi\in C_c^\infty(\Omega)$ $\forall u\in W_0^{k,p}(\Omega)$ $\langle\varphi,u\rangle := \int_\Omega u\varphi$.

φ 连续? $\forall u_n \xrightarrow{W_0^{k,p}(\Omega)} 0$. $\langle\varphi,u_n\rangle\to 0$.

$|\langle\varphi,u_n\rangle| \le \|u_n\|_{L^p}\|\varphi\|_{L^{p'}} \to 0$, $n\to\infty$

2. $C_c^\infty(\Omega)$ 在 $W^{-k,p'}(\Omega)$ 中稠.

$W_0^{k,p}(\Omega)$ 自反. $\left(W_0^{k,p}(\Omega)\right)^{**} = W_0^{k,p}(\Omega)$

$\forall f\in\left(W^{-k,p'}(\Omega)\right)^* = \left(W_0^{k,p}(\Omega)\right)^{**}$, $\exists u_f\in W_0^{k,p}(\Omega)$

$\langle f,v\rangle = \langle v,u_f\rangle$, $\forall v\in W^{-k,p'}(\Omega)$

若 f 作用在 $C_c^\infty(\Omega)$ 上都为0. $\forall\varphi\in C_c^\infty(\Omega)$ $\langle f,\varphi\rangle=0$

$\langle\varphi,u_f\rangle=0$ $u_f\in W_0^{k,p}(\Omega)$

$\Rightarrow u_f=0$. $f=0$ #

$W^{k,p}(\Omega) \Rightarrow (L^p)^{\sim}(\Omega)$ $\qquad L^p(\Omega) \longrightarrow (L^p)^{\sim}$ 自反

$u \leftrightarrow (u, D^{\alpha}u, D^{\alpha_2}u \cdots D^{\alpha_k}u)$ $\qquad \alpha_1 \quad |k|=1$

$u \leftarrow \quad u=1 \quad D^{\alpha_1}u=1$

$\tau: W^{k,p}(\Omega) \longrightarrow (L^p)^{\sim}(\Omega)$ $\qquad (L^p)^{\sim}$ 自反 $\Rightarrow \tau(W^{k,p}(\Omega))$ 自反

$u \longmapsto (u \cdots D^{\alpha_k}u)$ $\qquad (L^p)^{\sim}$ 闭子空间

$\tau: W^{k,p}(\Omega) \longrightarrow \tau(W^{k,p}(\Omega))$ 同构

闭

$W^{k,p}(\Omega) \supset W_0^{k,p}(\Omega)$

$(W^{k,p}(\Omega))^* \subset (W_0^{k,p}(\Omega))^*$

$C^{\infty}(\Omega) \subset (W^{k,p}(\Omega))^*$

$\overline{(C^{\infty}(\Omega))}^{(W^{k,p}(\Omega))^*} = \boxed{(W_0^{k,p}(\Omega))^*}$

Baire 定理及应用

Def. (Baire) 称 (X,τ) 是 Baire 空间，若任意可列多个稠密开子集之交仍然稠密。(任意可列多个无内点之闭集之并无内点)。

Thm. (Baire) 完备度量空间是 Baire 空间

证明：设 $\{O_n\}$ 稠密，开。往证 $\bigcap_n O_n$ 稠密。

$\forall B(x,r)$, $B(x,r) \cap \left(\bigcap_n O_n\right) \neq \phi$.

由 O_1 稠密，$B(x,r)\cap O_1 \neq \phi$. $\exists x_1 \in \overline{B_1} \subset B(x,r) \cap O_1$, $diam(B_1) < \frac{1}{2}$

又 O_2 稠密，$B_1 \cap O_2 \neq \phi$ $\exists x_2 \in \overline{B_2} \subset B_1 \cap O_2$, $diam(B_2) < \frac{1}{4}$.

... 一直做下去，$\{\overline{B_n}\}$ $diam(\overline{B_n}) \to 0$, $\overline{B_n} \downarrow$.

$\bigcap_n \overline{B_n} \neq \phi$. $\exists x \in \bigcap_n \overline{B_n}$, $x \in \bigcap_n O_n \cap B(x,r)$.

$\Rightarrow B(x,r) \cap \left(\bigcap_n O_n\right) \neq \phi$

Remark：一个度量空间不完备，能否是 Baire 空间？

可以。

$\phi(x) = \frac{x}{1+|x|}$, $x \in \mathbb{R}$. $d(x,y) = |\phi(x) - \phi(y)|$ d 不完备

$(\mathbb{R}, \tau_d) = (\mathbb{R}, \tau)$

Thm. 局部紧之 Hausdorff 空间也是 Baire 空间。X

$\forall x \in X$

U_x 是 x 之邻域

V_x $x \in V_x \subset\subset U_x$

证：$\{O_n\}$ 稠密，开。$\bigcap_n O_n$ 稠密。$\forall$ 开集 U

由 O_1 稠，$U \cap O_1 \neq \phi$. $\exists x_1 \in V_{x_1} \subset\subset U \cap O_1$

O_2 稠，$V_{x_1} \cap O_2 \neq \phi$ $\exists x_2 \in V_{x_2} \subset\subset U \cap O_2$

得到 $\{\overline{V_{x_n}}\}$, $V_{x_n} \downarrow$. $\overline{V_{x_n}}$ 紧。$\overline{V_{x_1}}$ 紧。

$\{\overline{V_{x_n}}\}$ 闭子族，$\bigcap_n \overline{V_{x_n}} \neq \phi$ $\exists x \in \bigcap_n \overline{V_{x_n}}$

$x\in(\bigcap_n O_n)\cap U$. $\quad (\bigcap_n O_n)\cap U\neq\phi$. #

Banach-Steinhaus Thm.

设 Z 是Banach空间，F 是赋范空间，$\{u_i\}_{i\in I}\subset B(Z,F)$，若对 $\forall x\in Z$.
$\sup_{i\in I}\|u_i(x)\|<+\infty$. 则

$$\sup_{i\in I}\|u_i\|<+\infty.$$

证明：$Z=\bigcup_{n=1}^{\infty}\{x\in Z\mid \sup_{i\in I}\|u_i(x)\|\le n\}=\bigcup_{n=1}^{\infty}F_n$.

F_n 闭. $\quad F_n=\bigcap_{i\in I}\{x\in Z\mid \|u_i(x)\|\le n\}$

$\boxed{\exists\ \mathring{F}_n\neq\phi}$ （若，对 $\forall n$，$\mathring{F}_n=\phi$，$Z=\bigcup_{n=1}^{\infty}F_n \Rightarrow Z$ 无内点 矛盾）

$\exists B(x,r)$. $\quad B(x,r)\subset F_n \quad \forall y\in B(x,r)$

$$\sup_{i\in I}\|u_i(y)\|\le n$$

$\forall z\in B(0,r)$ $\quad \|u_i(z+x)\|\le n \Rightarrow \|u_i(z)\|\le\|u_i(x)\|+n\le 2n$.

$$\|u_i\|\le\frac{2n}{r}.\quad \forall i\in I$$

$$\Rightarrow \sup_{i\in I}\|u_i\|\le\frac{2n}{r}.$$

Thm. Z 是 Banach，F 赋范. $(u_i)_{i\in I}\subset B(Z,F)$. $\sup_{i\in I}\|u_i\|=+\infty$

则 $\{x\in Z\mid \sup_{i\in I}\|u_i(x)\|=+\infty\}$ 是 Z 中稠密 G_δ 集. $(U\cdot G_\delta \Leftrightarrow U=\bigcap_n O_n)$

证明：$\{x\in Z\mid \sup_{i\in I}\|u_i(x)\|=+\infty\}=\bigcap_{n=1}^{\infty}\{x\in Z\mid \sup_{i\in I}\|u_i(x)\|>n\}=\bigcap_{n=1}^{\infty}O_n$.

O_n 稠密，开. $\quad \{x\in Z\mid \sup_{i\in I}\|u_i(x)\|\le n\}=(O_n)^c=F_n$ 闭 $\Rightarrow O_n$ 开

若 O_n 不稠密，$F_n=O_n^c$ 有内点，$\Rightarrow \sup_{i\in I}\|u_i\|<+\infty$ 矛盾. $\Rightarrow O_n$ 稠密，#

推论: Z. Banach, F赋范. $(u_n) \subset B(Z,F)$. 若 (u_n) 逐点收敛到 u.

则 $u \in B(Z,F)$ 且 $\|u\| \le \liminf \|u_n\|$

$u_n(x) \to u(x)$

证明: $u \in B(Z,F)$. u 有界性

$\sup_n \|u_n\| < +\infty$ $\qquad$ $\liminf_n \|u_n\| < +\infty$

$$\|u(x)\| = \lim_n \|u_n(x)\| \le \left(\liminf_n \|u_n\|\right) \|x\|$$

$$\Rightarrow \|u\| \le \liminf_n \|u_n\| < +\infty$$

推论: Z. Banach. $x_n \in Z$, $x_n \rightharpoonup x$ 则 $\|x\| \le \liminf \|x_n\|$

证明: $x_n \rightharpoonup x$ $\quad \forall f \in Z^*$. $\langle f, x_n \rangle \to \langle f, x \rangle$

$\langle x_n^{**}, f \rangle \to \langle x^{**}, f \rangle$

$\{x_n^{**}\} \subset Z^{**}$. $\{x_n^{**}\}$ 在 $\forall f \in Z^*$ 上有界. 则 $\sup_n \|x_n^{**}\| < +\infty$

$\sup_n \|x_n\| < +\infty$

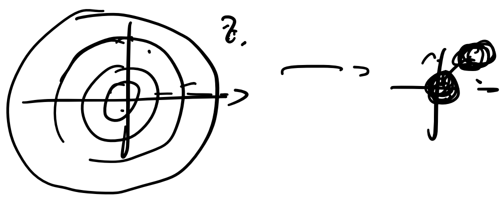
$$\|x^{**}\| \le \liminf_n \|x_n^{**}\|$$

$$\Rightarrow \|x\| \le \liminf_n \|x_n\|$$

Thm. (开映射定理) 设 Z, F. Banach, $u \in B(Z,F)$. 满射. 则 u 开映射.

证明: ① $B_F \subset \overline{u(cB_Z)}$, $\exists c > 0$

$$F = \bigcup_{n=1}^{\infty} u(B_n) = \bigcup_{n=1}^{\infty} \overline{u(B_n)} = \bigcup_{n=1}^{\infty} F_n$$



$\exists F_{n_0}$. $\mathring{F}_{n_0} \ne \phi$ $\quad$ ($\forall n$. $\mathring{F}_n = \phi \Rightarrow F$ 无内点, 矛盾.)

$\exists B(y,\eta)$, s.t. $B(y,\eta) \subset \overline{u(B_{n_0})}$.

$$\eta B_F \subset \overline{u(B_{n_0})} - y_0 \subset \overline{u(B_{2n_0})}$$

$\Rightarrow \exists c$. $B_F \subset \overline{u(cB_Z)}$.

② $B_F \subset \overline{u(cB_Z)} \subset u(2cB_Z)$.

$\forall y \in B_F$. $\exists x_0 \in c B_Z$, $\| y - u(x_0) \| \le \frac{1}{2}$.

$y_1 = 2[y - u(x_0)]$ $\quad \|y_1\| \le 1 \Rightarrow y_1 \in B_F$. $\exists x_1 \in c B_Z$.

s.t. $\|y_1 - u(x_1)\| \le \frac{1}{2}$

$\cdots$, $\{x_n\}$, $\{y_n\}$

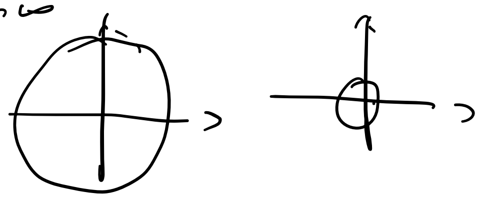
$y = \frac{1}{2} y_1 + u(x_0) = u(x_0) + \frac{1}{2} y_1$

$= u(x_0) + \frac{1}{2} u(x_1) + \frac{1}{4} y_2$

$= u\left(\sum_{k=0}^{n} \frac{x_k}{2^k}\right) + \underline{\frac{1}{2^{n+1}} y_{n+1}}$ $\quad (*)$

$n \to \infty$ $\quad y_n \in B_F$. $\|y_n\| \le 1$ $\quad \to 0$ $\quad \underline{\sum_{k=1}^{n}} \frac{x_k}{2^k} \to x$,

$\sum_{k=1}^{n} \frac{\|x_k\|}{2^k} \le C \sum_{k=1}^{n} \frac{1}{2^k} \le 2C$, $n \to \infty$



$\exists x \in B_{2C}$ $\quad \sum_{k=1}^{n} \frac{x_k}{2^k} \to x$

令 $n \to \infty$ $\quad y = u(x)$.

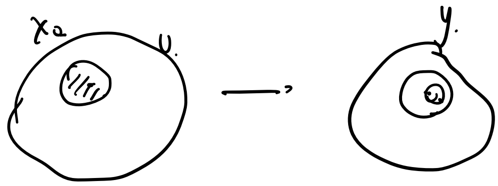
$\boxed{B_F \subset u(n B_Z)}$

$\eta B_F \subset u(\eta^n B_Z)$

③. 开映射. $\forall U$ 开. $u(U)$ 开集.

$\forall y \in u(U)$ $\quad x_0 \in U$. $u(x_0) = y$.

$\exists B(y, \eta) \subset u(U)$.



$\forall U \subset Z$. 开的 $\quad u(U)$ 开.

$\forall y_0 \in u(U)$. $\quad x_0 \in U$. $\quad u(x_0) = y_0$.

选择 $r > 0$. $\quad B(x_0, r) \subset U$, $\quad x_0 + B(0, r) \subset U$.

$y_0 + u(B(0, r)) \subset u(U)$

$u(B(0, r)) \supset B(0, \eta)$

$u(B(0, r)) \supset B(0, \eta)$

$B(y_0, \eta) \subset u(U)$

推论（逆映射定理）. $\mathcal{Z}, F$ Banach, $u \in \mathcal{B}(\mathcal{Z}, F)$. 双射 $\Rightarrow u^{-1} \in \mathcal{B}(F, \mathcal{Z})$

$$T: (\mathcal{Z}, \|\cdot\|_1) \to (\mathcal{Z}, \|\cdot\|_2)$$

$x \mapsto x$ 双. $\quad \|x\|_2 \le c\|x\|_1$

$\exists \tilde{c} \quad \|x\|_1 \le c\|x\|_2$.

Thm（闭图像定理）$\mathcal{Z}, F$ Banach 空间. $u \in \mathcal{L}(\mathcal{Z}, F)$, 则

$$u \text{ 连续} \iff G(u) \text{ 闭}$$

$$G(u) = \{(x, u(x)) \mid x \in \mathcal{Z}\}$$

证明: "$\Rightarrow$" $\forall x_n \to x$. $u(x_n) \to u(x)$. $(x, u(x)) \in G(u)$

"$\Leftarrow$" $G(u)$ 闭. $G(u)$ 是 $\mathcal{Z} \times F$ 上之闭的子空间. , $\mathcal{Z} \times F$.

$(x, y) \in \mathcal{Z} \times F$. $\quad \|(x, y)\|_{\mathcal{Z} \times F} = \|x\|_{\mathcal{Z}} + \|y\|_F$

$G(u)$ 是 Banach 空间.

$P: G(u) \to \mathcal{Z}$. $\quad (x, u(x)) \to x$ 双射.

$$\|x\| \le \|(x, u(x))\| = \|x\| + \|u(x)\|$$

P 连续. $\|P\| \le 1$.

P^{-1} 连续. $\quad \|(x, u(x))\| \le c\|x\| \Rightarrow \|u(x)\| \le c\|x\|$

$\Rightarrow u$ 连续.

eg 1. $\mathcal{Z}$ Banach. $T: \mathcal{Z} \to \mathcal{Z}^*$. 线性. 满足 $\langle Tx, y\rangle = \langle Ty, x\rangle$.

$\forall x, y \in \mathcal{Z}$, 则 T 连续

证明: $G(T)$ 闭. $\Leftarrow \begin{cases} x_n \to x \text{ in } \mathcal{Z}. \\ Tx_n \to f. \text{ in } \mathcal{Z}^* \end{cases} \Rightarrow f = Tx$

$$\langle Tx_n, y\rangle = \langle Ty, x_n\rangle \quad \forall y \in \mathcal{Z}.$$

$x_n \to x$. $\quad Tx_n \rightharpoonup f$ in $\mathcal{Z}^*$.

$$\langle Tx_n, y\rangle \to \langle f, y\rangle$$

$$\langle Ty, x_n\rangle \to \langle Ty, x\rangle$$

$\Rightarrow \langle f, y\rangle = \langle Ty, x\rangle = \langle Tx, y\rangle$.

$\Rightarrow \langle f - Tx, y\rangle = 0$. $\forall y \in \mathcal{Z}$. $\Rightarrow f = Tx$

eg2. $\mathcal{Z}, F$ Banach. $\quad u: \mathcal{Z} \to F$ 线性

(a) G是Hausdorff空间 $\quad v: F \to G$ 连续, 单射. $\quad u$连续 $\Leftrightarrow$ $v \circ u$连续

"$\Rightarrow$" 显然

"$\Leftarrow$" Graph(u) $\quad \begin{cases} x_n \to x \text{ in } \mathcal{Z} \\ u(x_n) \to y \text{ in } F \end{cases} \Rightarrow u(x) = y$.

$u(x_n) \to y$ in F.

v连续性 $\Rightarrow v \circ (u(x_n)) \to v(y)$

$v \circ u (x_n) \to v(u(x))$ $\quad \Rightarrow v(y) = v(u(x)) \Rightarrow y = u(x)$

(b) u连续 $\Leftrightarrow$ 当F上的拓扑换成更弱的Hausdorff拓扑, u连续

(F, τ_F) $\quad G = (F, \tau_{\mathcal{H}})$ $\quad \tau_{\mathcal{H}} \subset \tau_F$

$v: (F, \underline{\tau_F}) \to (F, \underline{\tau_{\mathcal{H}}})$ $\quad x \longmapsto x$, 单射, 连续

$v \circ u$连续 $\quad (\mathcal{Z}, \tau_{\mathcal{Z}}) \to (F, \tau_{\mathcal{H}})$ 连续

$\Rightarrow (\mathcal{Z}, \tau_{\mathcal{Z}}) \to (F, \tau_F)$ 连续

$\sigma(F, F^*)$

$u: (\mathcal{Z}, \|\cdot\|_{\mathcal{Z}}) \to \underline{(F, \sigma(F, F^*))}$ 连续, 则

$u: \underline{(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}}) \to (F, \|\cdot\|_F)}$ 连续.

证: 很显. $G(u)$ 闭.

$$\begin{cases} x_n \to x & \text{in } E \\ u(x_n) \to y & \text{in } F. \end{cases} \Rightarrow y = u(x).$$

$x_n \to x$ in E. $\Rightarrow$ $u(x_n) \to u(x)$ weakly

$\forall f \in F^*$, $\langle f, u(x_n) \rangle \to \langle f, u(x) \rangle$.

$\langle f, u(x_n) \rangle \to \langle f, y \rangle$

$\Rightarrow \langle f, u(x) \rangle = \langle f, y \rangle \quad \forall f \in F^*$

$\langle f, u(x) - y \rangle = 0$ $\qquad$ F^*. $x \in F$. $x \neq 0$. $\exists f \in F^*$ $f(x) \neq 0$.

$\Rightarrow u(x) - y = 0 \Rightarrow y = u(x)$

可数半范空间.

$\Omega \subset \mathbb{R}^n$ bdd. $\quad C(\bar{\Omega}) \quad \|f\| = \sup_{x \in \bar{\Omega}} |f(x)|$

$C(\Omega)$ $\qquad$ $f_n \to f$ in $C(\Omega)$

$\Leftrightarrow \forall K \subset\subset \Omega, \ \sup_{x \in K} |f_n(x) - f(x)| \to 0, \ n \to \infty$

$|f|_K := \sup_{x \in K} |f(x)|$, $K \subset\subset \Omega$. $\qquad p(x) = 0 \Leftrightarrow x = 0$ ✗

$\boxed{|\cdot|_K}$ 半范. ~~范数~~ $\qquad |\cdot|_K$.

$K \subset\subset \Omega \quad \{|\cdot|_K\} \to \tau$.

$f \neq 0$. $|f|_K = 0$

$\Omega - K$

Def. 设线性空间. $p: E \to [0, +\infty)$

(1) $p(x) \geq 0$. $\qquad$ (2) $p(\lambda x) = |\lambda| p(x)$, $\forall x \in E, \lambda \in \mathbb{K}$.

(3) $p(x+y) \leq p(x) + p(y)$ $\qquad$ (半范)

Remark: $d(x,y) := p(x-y)$. $B(x,\varepsilon) = \{y \in \tau \mid d(x,y) < \varepsilon\}$ => 拓扑. Tp.

Tp Hausdorff <=> p 是一个范数

$p(x) = 0 \Rightarrow x = 0$ $x \neq 0$. $p(x) = 0$. x. 0

Def. $(p_i)_{i\in I}$ 是 τ 上一族半范，定义 $(p_i)_{i\in I}$ 诱导之拓扑 τ.

$$O \in \tau \iff O = \bigcup_{\alpha\in\Lambda} B_{g J_\alpha}(x_\alpha, r_\alpha)$$

α 指标集. $J_\alpha \subset I$. 有限.

$$B_{g J_\alpha}(x_\alpha, r_\alpha) := \{y \in \tau \mid \max_{i\in J_\alpha} p_i(x_\alpha - y) < r_\alpha\}$$

τ 是一个拓扑. ϕ. $x \in \tau$, 且任意并封闭，只需要验证有限交封闭.

$$\bigcap_{i=1}^{n} \bigcup_{\alpha\in\Lambda_i} B_{g J_\alpha}(x_\alpha, r_\alpha) \quad \Lambda_1, \cdots \Lambda_n \text{ 指标集.}$$

$\forall \alpha_i \in \Lambda_i$ $\bigcap_{i=1}^{n} B_{g J_{\alpha_i}}(x_{\alpha_i}, r_{\alpha_i})$

$\forall x \in \bigcap_{i=1}^{n} B_{g J_{\alpha_i}}(x_{\alpha_i}, r_{\alpha_i})$

$x \in B_{g J_{\alpha_i}}(x_{\alpha_i}, r_{\alpha_i}) \iff \forall p \in g J_{\alpha_i}$ $p(x - x_{\alpha_i}) < r_{\alpha_i}$.

考虑 $B_{g J_{\alpha_1} \cup \cdots J_{\alpha_n}}(x, \varepsilon)$. ε 足够小. $B_{g J_{\alpha_1}\cup\cdots J_{\alpha_n}}(x,\varepsilon) \subset \bigcap_{i=1}^{n} B_{g J_{\alpha_i}}(x_{\alpha_i}, r_{\alpha_i})$

$\forall y \in B_{g J_{\alpha_1}\cup\cdots J_{\alpha_n}}(x,\varepsilon)$ $\forall p \in g J_{\alpha_i}$ $p(x - x_{\alpha_i}) < r_i$.

$$p(y - x_{\alpha_i}) \leq p(y-x) + p(x - x_{\alpha_i}) \leq \varepsilon + p(x - x_{\alpha_i}) < r_i.$$

$\bigcap_{i=1}^{n} B_{g J_{\alpha_i}}(x_{\alpha_i}, r_{\alpha_i})$ 开集.

Remark. $\{p_i\}_{i\in I}$. $g J \subset I$. 有限 $B_{g J}(x, \varepsilon)$

若 $\{p_i\}_{i\in I}$ 可定向. $\forall p_1, p_2$, $\exists p_3$ s.t. $p_3 \geq \max\{p_1, p_2\}$

$g J \subset I$ $p \in I$ $B_{p_1 \cup p_2}(x, \varepsilon) \supset B_{p_3}(x, \varepsilon)$. $B_{p_i}(x, \varepsilon)$.

$\boxed{\forall x \neq 0.\ \exists p_i(x) \neq 0}$

Remark. $(p_i)_{i\in I}$ 诱导的拓扑 Hausdorff 的 $\Leftrightarrow$ p_i 可分离 $\boxed{\forall x\neq y,\ \exists p_i(x)\neq p_i(y)}$

$p_i=\|f\|$

"$\Rightarrow$" $\forall x\in E$, $x\neq 0$. $\exists B_{\mathcal{J}}(x,\varepsilon)\not\ni 0$

$\forall p\in\mathcal{J}$, $p(x)\geq\varepsilon$, $p(x)\neq 0$.

"$\Leftarrow$" $\forall y\neq 0$, $\exists p_i(x)\neq 0$.

$\forall x\neq y$, $x,y\in E$, $x-y\neq 0$. $\exists p_i$ s.t. $p_i(x-y)\neq 0$.

当ε充分小, $B_{p_i}(x,\varepsilon)\cap B_{p_i}(y,\varepsilon)=\phi$.

$p_i(z-x)\le\varepsilon$, $p_i(z-y)\le\varepsilon$ $\Rightarrow$ $p_i(x-y)\le 2\varepsilon<p_i(x-y)$ 矛盾.

Def (拓扑向量空间). 称E是数域$\mathbb{K}$上的向量空间, τ是E上的拓扑. 若

$$\Phi:\begin{cases} E\times E\to E\\ (x,y)\mapsto x+y\end{cases}\qquad \Psi:\begin{cases}\mathbb{K}\times E\to E\\ (\lambda,x)\mapsto\lambda x\end{cases}$$

连续, 则称E是拓扑向量空间.

Thm. 设 $(p_i)_{i\in I}$ 是向量空间E上的半范数族, 那么由(p_i)诱导的拓扑与E的线性结构相容, 并且该拓扑是使$\{p_i\}$都连续的最弱拓扑.

pf. (p_i)连续在τ下.

$\forall x\in B_{p_i}(x_0,\varepsilon)$ $|p_i(x)-p_i(x_0)|\le p_i(x-x_0)<\varepsilon$. $\Rightarrow$ p_i连续.

τ最弱. 假设τ'是使p_i都连续的拓扑. 往证 $\tau\subset\tau'$

$\mathcal{N}_\tau(0)\subset\mathcal{N}_{\tau'}(0)$. 任取 $V\in\mathcal{N}_\tau(0)$ 找 $U\in\mathcal{N}_{\tau'}(0)$. s.t. $U\subset V$.

存在有限集 $\mathcal{J}$. s.t. $B_{\mathcal{J}}(0,\varepsilon)\subset V$.

$B_{\mathcal{J}}(0,\varepsilon)=\bigcap_{i\in\mathcal{J}}B_{p_i}(0,\varepsilon)$, p_i在τ'下连续. $B_{p_i}(0,\varepsilon)\in\mathcal{N}_{\tau'}(0)$

$B_{\mathcal{J}}(0,\varepsilon)\in\mathcal{N}_{\tau'}(0)$ #

Thm. 可半赋范空间是局部凸的，即在零点有一族凸的邻域基

Pf: q_J 邻域 $\{B_{q_J}(0,\varepsilon)\}$

$(E,\|\cdot\|)\to(F,\|\cdot\|)$

$\exists c$ s.t. $\|u(x)\|\le c\|x\|$

Thm. 设 $(E,(p_i)_{i\in I})$, $(F,(q_j)_{j\in J})$, $u:E\to F$ 线性

u 连续 $\iff$ 对 $\forall J$ 的有限子集 J'，存在 I 的有限子集 I' 及 $c>0$ s.t.

$$\max_{j\in J'} q_j(u(x)) \le c\max_{i\in I'} p_i(x)\qquad \forall x\in E.$$

Pf: "$\Rightarrow$" u 连续，$u^{-1}\left(B_{\max_{j\in J'} q_j}(0,1)\right)$ 是 0 邻域，$\exists I'$.

$$B_{\max_{i\in I'}p_i}(0,r)\subset u^{-1}\left(B_{\max_{j\in J'}q_j}(0,1)\right)$$

即当 $\max_{i\in I'}p_i(x)<r$ 时，$\max_{j\in J'}q_j(u(x))<1$.

$\max_{i\in I'} p_i(\cdot)<r$

$\forall x\in E$, $\delta>0$, $\max_{j\in J'} q_j\left(u\left(\frac{r}{\max_{i\in I'}p_i(x)+\delta}x\right)\right)<1$

$$\max_{j\in J'} q_j(u(x))<\frac{\max_{i\in I'}p_i(x)+\delta}{r},\quad \delta\to 0,\quad c=\frac{1}{r}$$

"$\Leftarrow$" $\forall V\in N(0)$, 存在 $J'\subset J$ 有限子集 及 $\varepsilon>0$ s.t. $B_{\max_{j\in J'}q_j}(0,\varepsilon)\subset V$.

$\exists I'\subset I$ 及 $c>0$. $r=\frac{\varepsilon}{c}$ 时. $B_{\max_{i\in I'}p_i}(0,r)\subset u^{-1}\left(B_{\max_{j\in J'}q_j}(0,\varepsilon)\right)$

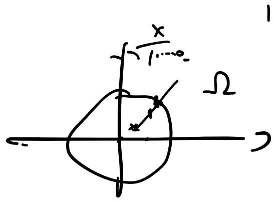
$u^{-1}(V)\in N(0)$ $\Rightarrow$ u 连续.

二、局部凸拓扑向量空间 $\Rightarrow$ 可半赋范

$\frac{x}{10}$

Minkowski 泛函 Ω

$0\in\Omega$ 开集, $p_\Omega(x)=\inf\{\lambda>0 \mid \frac{x}{\lambda}\in\Omega\}$



Ω 平衡 $\quad\forall x\in\Omega$, $|\lambda|\le 1$ $\quad\lambda x\in\Omega$

$\Rightarrow$ $\Omega=\{x: p_\Omega(x)<1\}$

$x\in\Omega \iff p_\Omega(x)<1$

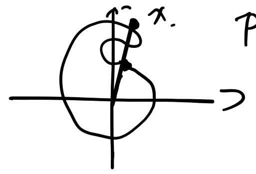
$x\in\partial\Omega \quad p_\Omega(x)=1$

$p_\Omega^{-1}((-\varepsilon,\varepsilon)) = \varepsilon p_\Omega^{-1}((-1,1)) = \varepsilon\Omega$

$\forall x\in\Omega$. $|\lambda|\le 1$ $\lambda x\in\Omega$.

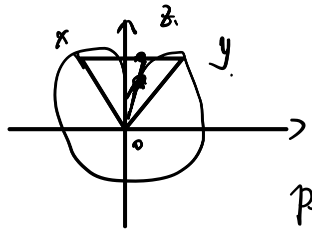
Ω 开集. $\Rightarrow$ $\varepsilon\Omega$ 开集. $\Rightarrow$ p_Ω 连续

Ω 凸集 $\Rightarrow$ p_Ω 半范. 三角不等式.



$p_\Omega(x) < 1$

$\ne 1$.



$p_\Omega(x) = 1$

$p_\Omega(y) = 1$

$p_\Omega(z) > 1$

$z = tx + (1-t)y$.

$p_\Omega(z) \le t\, p_\Omega(x) + (1-t)\, p_\Omega(y)$

≤ 1. 矛盾.

局部平衡凸的开邻域基

Lem. 拓扑向量空间的局部有平衡的开邻域基

pf. 对于 $\forall V\in\mathcal{N}(0)$ 存在 $U\subset V$. 平衡 $U\in\mathcal{N}(0)$.

$\Phi: \mathbb{K}\times X\to X$ $(\lambda,x)\mapsto\lambda x$, $\Phi(0,0)=0$

$\exists\, B_{\mathbb{K}}(0,\delta)$ 及 $V'\in\mathcal{N}(0)$. s.t. $\Phi(B_{\mathbb{K}}(0,\delta)\times V')\subset V$

即 $\forall\, 0<|\lambda|<\delta$. $\lambda V'\subset V$.

令 $U = \bigcup_{|\lambda|<\delta}\lambda V'\subset V$. 平衡 $\forall\mu\in\mathbb{K}$, $|\mu|\le 1$.

$\mu U\subset U$. $\mu U = \bigcup_{|\lambda|<\delta}\mu\lambda V'$ $|\mu|\le 1$. $|\mu\lambda|<\delta$.

$\subset \bigcup_{|\lambda|<\delta}\lambda V' = U$.

$\Rightarrow$ U 平衡. $U\in\mathcal{N}(0)$ 开.

Thm. 局部凸拓扑向量空间 局部有凸的平衡开邻域基.

pf. $\{U_\alpha\}_{\alpha\in\Lambda}$. $\{\mathring{\mathrm{conv}}(U_\alpha)\}_{\alpha\in\Lambda}$.

$\forall V\in\mathcal{N}(0)$. $\exists\, A\in\mathcal{N}(0)$. $A\subset V$. A 凸

$\exists\, B\in\mathcal{N}(0)$. s.t. $B\subset A$, B 平衡开邻域.

$\mathrm{conv}(B) = \left\{\sum_{i=1}^n\lambda_i b_i ;\ 0\le\lambda_i\le 1,\ \sum_{i=1}^n\lambda_i = 1,\ b\in B\right\}$

A凸 $B\subset A$. $\mathrm{conv}(B)\subset A$. $\mathrm{conv}(B)$ 平衡

$(\mathrm{conv}(B))^\circ$: 开的, 凸的, 平衡的. $(\mathrm{conv}(B))^\circ \subset V$.

Def. (Minkowski). Ω 开, 凸, 平衡. $\Omega \in \mathcal{N}(0)$, $p_\Omega : X \to \mathbb{R}$

$$p_\Omega(x) := \inf\{\lambda > 0 : \tfrac{x}{\lambda} \in \Omega\}.$$

prop. (1) $x \in \Omega \iff p_\Omega(x) < 1$. $\Leftarrow$ Ω 平衡

(2) $\Omega_1 \subset \Omega_2 \Rightarrow p_{\Omega_2} \le p_{\Omega_1}$

(3) $\Omega_3 = \Omega_1 \cap \Omega_2$, $p_{\Omega_3} \ge \max\{p_{\Omega_1}, p_{\Omega_2}\}$.

保证 $\{p_\Omega\}$ 定向.

Thm. τ 局部凸. $(p_\Omega)_\Omega$ 一族半范, Ω 是取遍原点的凸平衡开邻域, 则 p_Ω 诱导的拓扑 τ' 与原拓扑 τ 一致.

pf: "$\Rightarrow$" $\tau' \subset \tau$.

p_Ω 连续, 在 τ 下. $p_\Omega^{-1}(-\varepsilon, \varepsilon) = \varepsilon\, p_\Omega^{-1}(-1,1) = \varepsilon\Omega$ 开集.

"$\Leftarrow$" $\tau \subset \tau'$

$\forall\, O \in \mathcal{N}_\tau(0)$, $\exists\, U \in \mathcal{N}_{\tau'}(0)$. $U \subset O$.

$O \in \mathcal{N}_\tau(0)$ $\exists\, \Omega \subset O$, 凸的, 平衡的, 开的.

$$\Omega = \{x \in X : p_\Omega(x) < 1\} \in \mathcal{N}_{\tau'}(0)$$

$U = \Omega \subset O$. $\Omega \in \mathcal{N}_{\tau'}(0)$. $\Rightarrow \tau \subset \tau'$. #

eg.1. $C^\infty([0,1])$ $\quad p_i(f) = \sup_{x \in [0,1]} |f^{(i)}(x)|$.

$\forall f \neq 0$. $p_0(f) = \sup_{x\in[0,1]} |f(x)| \neq 0 \Rightarrow \{p_i\}$ 可分离.

$f_n \to f \iff \forall i.\ f_n^{(i)} \rightrightarrows f^{(i)}$ in $[0,1]$

$\{p_i\}$ 诱导拓扑 可度量化. $d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{p_k(x-y)+1}$

$C^\infty([0,1]^d)$. $p_\alpha(f) = \sup_{x\in[0,1]} |D^\alpha f(x)|$, $\{p_\alpha\}$ 可分离, 可数多个 ⇒ 可度量化.

eg2. X 局部紧的 Hausdorff 空间, $C(X,\mathbb{K})$

设 $K\subset\subset X$, $p_K(f) = \sup_{x\in K}|f(x)|$

$\{p_K\}$ 半范, 可分离, $f_n \rightrightarrows f \iff f_n \rightrightarrows f$ in K, $\forall K\subset\subset X$

定义 $K_i \uparrow$, $\bigcup_{i=1}^{\infty} K_i = X$ $\{p_{K_i}\}$ 可定向 ⇒ 可度量化

eg3. $S(\mathbb{R}^n) = \{ f\in C^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty \}$

$(\|\cdot\|_{\alpha,\beta})$ 半范, 可数多个 ⇒ 可度量化, 不可赋范.

eg4. $\Omega\subset\mathbb{R}^n$, bdd, $0<p<\infty$, $L^p_{loc}(\Omega)$ $L^p(\Omega)$

$\forall K\subset\subset\Omega$, $f\in L^p(K)$ $1\le p\le\infty$. $\|\cdot\|$

$0<p<1$. 度量

$$N_{p,K}(f) = \left(\int_K |f|^p\right)^{\frac{1}{p}}$$

$p\geq 1$ 时. $N_{p,K}$ 半范数 ⇒ 局部凸空间

$0<p<1$. $N_{p,K}$ ~~半范数~~. 不满足三角不等式. 非局部凸

$(Z,\|\cdot\|) \to (Z,\{p_K\})$

$u: X\to X$ 连续 $\iff$ $\forall p_K$ $\exists c$. $|p_K(x)| \le c\|x\|$

$(Z,p_K) \to (Z,\|\cdot\|)$

$u: X\to X$ 连续, $\exists$ 有限个 $p_1\cdots p_k$. s.t. $\|x\| \le c \max_{i=1\cdots k} |p_i(x)|$

可定向, $\|x\|\le c\,|p_K(x)|$ (Z,p_K) 是 Hausdorff

⇒ $(Z,\|\cdot\|)$ 和 (Z,p_K) 等价. 要使 $(Z,\|\cdot\|)$ Hausdorff p_K 是范数

Def. $\mathfrak{X}$ 是 Hausdorff 拓扑空间，$\mathfrak{X}^*$ 可分点. $\mathfrak{X}^*$ 诱导之一致拓扑称为弱拓扑.

prop. $\mathfrak{X}$ 赋范, $\{x_n\} \subset \mathfrak{X}$.

(i) $x_n \to x$ in $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ $\Longleftrightarrow$ $\forall f \in \mathfrak{X}^*$. $\langle f, x_n \rangle \to \langle f, x \rangle$.

(ii) $x_n \to x \implies x_n \to x$ in $\sigma(\mathfrak{X}, \mathfrak{X}^*)$

$|f(x_n) - f(x)| \le \|f\| \, \|x_n - x\| \to 0$.

(X, τ) τ 离散. $x_n \in X$ $x_n \to x$, $\exists N$. s.t. $n > N$ 有 $x_n = x$.

$\tau = \{\phi, X\}$. x_n 收敛.

(iii) $x_n \to x$ in $\sigma(\mathfrak{X}, \mathfrak{X}^*)$, $\|x_n\|$ 有界 且 $\|x\| \le \liminf \|x_n\|$

$\forall f \in \mathfrak{X}^*$, $f(x_n) \to f(x)$. $T: \mathfrak{X} \to \mathfrak{X}^{**}$ $x \mapsto \hat{x}$. $\langle \hat{x}, f \rangle = \langle f, x \rangle$

$\hat{x}_n$ 在 $\mathfrak{X}^*$ 上逐点有界. $\forall f \in \mathfrak{X}^*$. $\hat{x}_n(f) = f(x_n) \to f(x)$

$\sup_n \|\hat{x}_n\|_{\mathfrak{X}^{**}} < +\infty \implies \sup_n \|x_n\| < +\infty$.

(iv) $x_n \to x$ in $\sigma(\mathfrak{X}, \mathfrak{X}^*)$, $f_n \to f$ in $\mathfrak{X}^*$. $\implies \langle f_n, x_n \rangle \to \langle f, x \rangle$

$$|\langle f_n, x_n \rangle - \langle f, x \rangle| \le \underbrace{\|f_n - f\| \, \|x_n\|}_{\to 0} + \underbrace{|\langle f, x_n - x \rangle|}_{\to 0}$$

Prop. 若 $\mathfrak{X}$ 有限维，则 $(\mathfrak{X}, \sigma(\mathfrak{X}, \mathfrak{X}^*)) = (\mathfrak{X}, \|\cdot\|)$

Pf. $\tau_{\|\cdot\|} \subset \sigma(\mathfrak{X}, \mathfrak{X}^*)$

$\forall B(0,r)$. 存在弱邻域 $V \subset B(0,r)$

$f_1(x_1 e_1 + \cdots + x_n e_n) = x_1 \quad \cdots \quad f_n(x_1 e_1 + \cdots + x_n e_n) = x_n$. $f_1 \cdots f_n \in \mathfrak{X}^*$

$\{y : \max_{1 \le i \le n} |f_i(y)| < \varepsilon\} \subset B(0,r)$

$|y_i| < \varepsilon \implies |y| \le C(\varepsilon) \to 0$, $\varepsilon \to 0$

remark. 有限维空间中弱拓扑等价于强拓扑.

Example. Z 赋范. $\dim Z = +\infty$ $S = \{x \in Z : \|x\| = 1\}$. 但 S 在 $\sigma(Z,Z^*)$ 下闭包

$$\overline{S}^{\sigma(Z,Z^*)} = B_Z. \text{（闭单位球）}$$

$L^2([0,1])$ $\sin 2n\pi x \to 0$

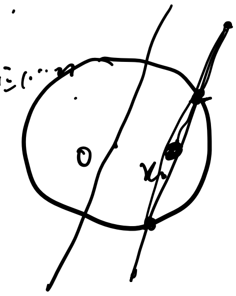
Pf. $\forall\, x_0 \in Z$, $\|x_0\| < 1$, $x_0 \in \overline{S}^{\sigma(Z,Z^*)}$.

即证任意 x_0 在 $\sigma(Z,Z^*)$ 下之邻域 V, $V \cap S \neq \phi$

$V = \{x \in Z : |f_i(x - x_0)| < \varepsilon\}$ $i = 1, \dots, k$, $f_i \in Z^*$.

$\forall\, t \in \mathbb{R}$, $x_0 + ty \in V$

Claim# 存在 $y_0 \in Z$, $y_0 \neq 0$, s.t. $\langle f_i, y_0 \rangle = 0$, $\forall\, i = 1, \dots, k$



考虑 $\varphi : Z \to \mathbb{R}^k$.

$$\varphi(x) = (f_1(x), f_2(x), \dots, f_k(x))$$

$\varphi(x) = 0 \Rightarrow x = 0$ φ 单 $\Rightarrow \dim Z \le \dim(\mathbb{R}^k) = k$. 矛盾

$\{x_0 + ty_0 \mid t \in \mathbb{R}\} \subset V$. $x_0 + ty_0$. $|f_i(x_0 + ty_0 - x_0)| = 0 < \varepsilon$

$\exists\, t_0$ s.t. $\|x_0 + t_0 y_0\| = 1$, $\|x_0\| < 1$. $t \to \infty$ $\|x_0 + ty_0\| \ge |t|\,\|y_0\| - \|x_0\| \to \infty$

$\Rightarrow V \cap S \neq \phi$.

$$\overline{S}^{\sigma(Z,Z^*)} \supset B_Z \supset S$$

$\forall\, x \notin B_Z$, $\{x\}$ 紧, B_Z 闭, $\{x\} \cap B_Z = \phi$, $\exists\, f \in Z^*$,

$$f(x) < \alpha < f(B_Z)$$

存在 x 的弱邻域 V s.t. $V \cap B_Z = \phi$ B_Z^c 在弱拓扑下开 $\Rightarrow B_Z$ 弱闭

Example. $U = \{x \in Z : \|x\| < 1\}$ 在弱拓扑下不开. $(U)^{\circ\, \sigma(Z,Z^*)} = \phi$.

$U^c = \{x \in Z : \|x\| \ge 1\}$ $\overline{(U^c)}^{\sigma(Z,Z^*)} = Z$.

$\Rightarrow (\mathring{U})^{\sigma(Z,Z^*)} = \phi$

problem: 弱收敛 与 强收敛

强收敛 $x_n \to x$ $\Leftrightarrow$ 弱收敛 $x_n \to x$.

$(x_n) \in l_1 \Leftrightarrow \sum_n |x_n| < +\infty$

收敛性相同 与 拓扑相同. X 不可数.

(X, τ), τ离散. $O \in \tau \Leftrightarrow O \in 2^X$ $\quad x_n \to x \Leftrightarrow \exists N\ n>N$ 时 $x_n = x$

τ 可数余拓扑. $O \in \tau \Leftrightarrow O^c$ 至多可数

$x_n \to x \Leftrightarrow \exists N$ s.t. $n>N$ 时, $x_n = x$

Thm (Mazur). $C \subset E$. convex. C 在 $\sigma(E, E^*)$ 闭 $\Leftrightarrow$ 强拓扑下.

"$\Rightarrow$" 弱闭 $\to$ 强闭

"$\Leftarrow$" 强闭, $\overline{C}^{\sigma(E,E^*)} \supsetneq C$. $\exists x \in \overline{C}^{\sigma(E,E^*)} \setminus C$.

$\{x\}$ 紧, C 闭凸集, $\exists f \in E^*$. $f(x) < \alpha < f(y)$. $\forall y \in C$

Cor. $(x_n) \subset E$, $x_n \rightharpoonup x$, 则 存在 y_n 是 (x_n) 之有限凸组合. s.t. $y_n \to x$

Pf. $C = \operatorname{conv}\left(\bigcup_{n=1}^{\infty} \{x_n\}\right)$ 凸. $x_n \to x \Rightarrow x \in \overline{C}^{\sigma(E,E^*)} = \overline{C}^{\|\cdot\|}$

$\exists y_n \in C$. s.t. $y_n \to x$

$\operatorname{conv} A = \left\{ \sum_{i=1}^{n} \lambda_i a_i \mid a_i \in A, 0 < \lambda_i < 1\ \sum_{i=1}^{n} \lambda_i = 1 \right\}$.

Thm. E, F Banach. $T: E \to F$ 线性

连续

$T: (E, \|\cdot\|) \to (F, \|\cdot\|)$ 连续 $\Leftrightarrow$ $\underline{T: (E, \sigma(E, E^*)) \to (F, \sigma(E, E^*))}$

Pf: $\Rightarrow$ $x_n \to x$ in $\sigma(E, E^*)$ $\qquad T: E \to F$. $\qquad f \circ T: E \to \mathbb{R}$

$Tx_n \to Tx$ in $\sigma(F, F^*)$ $\qquad f: F \to \mathbb{R}$ $\qquad f \circ T \in E^*$.

$\forall f \in F^*$. $f(Tx_n) = f \circ T(x_n) \to f \circ T(x) = f(Tx)$

$\Leftarrow$ $G(T)$ 闭，在 $(\mathcal{X}\times F,\ \sigma(\mathcal{X},\mathcal{X}^*)\times\sigma(F,F^*))$

$x_n\to x$ in $\sigma(\mathcal{X},\mathcal{X}^*)$. $Tx_n\to Tx$ in $\sigma(F,F^*)$

$\sigma(\mathcal{X},\mathcal{X}^*)\times\sigma(F,F^*) = \sigma(\mathcal{X}\times F,\ (\mathcal{X}\times F)^*)$

$G(T)$ 在 $(\mathcal{X}\times F,\ \sigma(\mathcal{X}\times F,(\mathcal{X}\times F)^*))$ 闭.

$\in G(T)$

$G(T)$ 凸. $(x, Tx)\in G(T)$. $(y,Ty)\in G(T) \Rightarrow (\lambda x+(1-\lambda)y,\ \lambda Tx+(1-\lambda)Ty)$

$\Rightarrow$ $G(T)$ 在 $(\mathcal{X}\times F,\ \|\cdot\|_{\mathcal{X}\times F})$ 闭 $\|\cdot\|_{\mathcal{X}\times F}=\max\{\|\cdot\|_{\mathcal{X}},\ \|\cdot\|_F\}$

$\Rightarrow$ T 连续 $(\mathcal{X},\|\cdot\|)\to(F,\|\cdot\|)$

弱*拓扑

Def. $\mathcal{X}$ 是赋范空间, $\tau:\mathcal{X}\to\mathcal{X}^{**}$ $x\mapsto\hat{x}$, $\hat{x}(f)=\langle f,x\rangle$, $\forall f\in\mathcal{X}^*$.
则 $\{|\hat{x}|\}_{x\in\mathcal{X}}$ 是 $\mathcal{X}^*$ 上一族半范数 诱导之拓扑称为 $\mathcal{X}^*$ 上之弱*拓扑. 记为 $\sigma(\mathcal{X}^*,\mathcal{X})$

$\forall x_1\cdots x_n\in\mathcal{X}$. $B_{x_1\cdots x_n}(f,\varepsilon)=\{g\in\mathcal{X}^*\mid |f(x_i)-g(x_i)|<\varepsilon,\ i=1\cdots n\}$.

Remark. $\mathcal{X}^*$. $\|\cdot\|_{\mathcal{X}^*}\supset\sigma(\mathcal{X}^*,\mathcal{X}^{**})\supset\sigma(\mathcal{X}^*,\mathcal{X})$

Banach-Alaoglu Th. $\mathcal{X}^*$. $S=\{x\in\mathcal{X}^*;\ \|x\|=1\}$ 不是 $\|\cdot\|$ 紧的,
是弱*紧的.

prop. 弱*拓扑是Hausdorff.

$\{p_i\}_{i\in I}\Rightarrow\tau$. τ Hausdorff $\Leftrightarrow$ $\forall x\neq0$. $\exists p_i$ $p_i(x)\neq0$

$x\in\mathcal{X}$. $\{|\hat{x}|\}_{x\in\mathcal{X}}$. $\forall f\in\mathcal{X}^*$. $f\neq0$. $\exists x.$ s.t. $f(x)\neq0$. 又 $f\neq0$

$\Rightarrow\{|\hat{x}|\}_{x\in\mathcal{X}}$ 可分点.

prop. $(f_n)\subset\mathcal{X}^*$.

(i) $f_n\xrightarrow{*}f$ in $\sigma(\mathcal{X}^*,\mathcal{X})$ $\Leftrightarrow$ $\langle f_n,x\rangle\to\langle f,x\rangle$, $\forall x\in\mathcal{X}$.

(ii) If $f_n \to f$, $\Rightarrow f_n \rightharpoonup f$ in $\sigma(E^*, E^{**})$

If $f_n \rightharpoonup f$ in $\sigma(E^*, E^{**}) \Rightarrow f_n \xrightarrow{*} f$ in $\sigma(E^*, E)$

$\forall \varphi \in E^{**}$ $\quad \varphi(f_n) \to \varphi(f)$. $\quad \forall x \in E$. $\quad \hat{x} \in E^{**}$ $\quad \hat{x}(f_n) \to \hat{x}(f)$

(iii). $f_n \xrightarrow{*} f$ in $\sigma(E^*, E)$, $\|f_n\|$ 有界, 且 $\|f\| \le \liminf \|f_n\|$

$\forall x \in E$. $\langle f_n, x\rangle \to \langle f, x\rangle$. $\{f_n\}$ 在E上逐点有界 $\Rightarrow \sup_n \|f_n\| < +\infty$

(iv). $f_n \xrightarrow{*} f$ in $\sigma(E^*, E)$, $x_n \to x$ in E. $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

$$|\langle f_n, x_n\rangle - \langle f, x\rangle| \le \underbrace{|\langle f_n, x_n\rangle - \langle f_n, x\rangle|}_{\to 0} + \underbrace{|\langle f_n, x\rangle - \langle f, x\rangle|}_{\to 0}$$

$x_n \rightharpoonup x$ $\not\Rightarrow$ $\langle f_n, x_n\rangle \to \langle f, x\rangle$

Remark. E $\dim(E) < +\infty$ $\quad J: E \to E^{**}$, $x \mapsto \hat{x}$.

$\dim(E^{**}) = \dim(E)$. $\quad E \subset E^{**}$ $\quad \Rightarrow E = E^{**}$.

E^{**}. $\to \mathbb{R}$. $\qquad$ H.

prop. $\varphi: E^* \to \mathbb{R}$ 线性, 在弱*拓扑下连续, 则存在 $x_0 \in E$. s.t.

$\varphi(f) = \langle f, x_0\rangle$, $\forall f \in E^*$. $\quad$ ② x_0. $\quad E \subset E^{**}$

Pf: $\quad E \subset H$. $\qquad \forall x \in E$, $\hat{x}$.

$H \subset E^{**}$. $\qquad \varphi: (E^*, \text{弱}^*) \to \mathbb{R}$, 连续 $\Rightarrow \varphi \in E^{**}$, $\Rightarrow H \subset E^{**}$.

$E \subset H \subset E^{**}$.

Lem. X 向量空间, $\varphi, \varphi_1, \cdots, \varphi_k$ 是 X 上 $(k+1)$ 个线性泛函, 满足

若 $\varphi_i(v) = 0$, $\forall i = 1, 2, \cdots, k$, 则 $\varphi(v) = 0$

$\Rightarrow$ 存在 $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$. $\quad$ s.t. $\varphi = \sum_{i=1}^{k} \lambda_i \varphi_i$

Pf: 考虑: $F: X \to \mathbb{R}^{k+1}$ $\quad u \mapsto [\varphi(u), \varphi_1(u), \cdots, \varphi_k(u)]$.

$(1, 0, \cdots, 0) \notin R(F)$

为了证明这个结论 $u(0,\dots,0)$ 和 $\mathbb{R}(F)$

$\lambda_0 x_0 + \lambda_1 x_1 + \dots + \lambda_k x_k = 0$

$u \to \mu u$

$\lambda_0 < \alpha < [\lambda_0 \varphi(u) + \dots + \lambda_k \varphi(u)]$ μ 对 $\forall u \in X$ 成立.

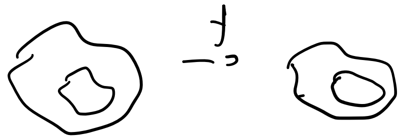
$\Rightarrow$ $\lambda_0 \varphi(u) + \dots + \lambda_k \varphi(u) = 0$ 于. 考虑 μu. $\mu \in \mathbb{R}$

$\Rightarrow$ $\lambda_0 < 0$.

$$\varphi(u) = -\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_0} \varphi_i(u).$$

Pf of prop. φ 在弱*拓扑下连续. $\exists V \in \mathcal{N}(0)$ (弱*拓扑)

$$|\varphi(f)| < 1 \quad \forall f \in V.$$



假设 $V = \{f \in E^* : |\langle f, x_i \rangle| < \varepsilon,\ i = 1, 2, \dots, k\}$

$\langle f, x_i \rangle = 0$, $\forall i = 1 \dots k$. $\Rightarrow$ $\varphi(f) = 0$

$|\langle f, x_i \rangle| < \delta \varepsilon$. $\Leftrightarrow |\langle \frac{f}{\delta}, x_i \rangle| < \varepsilon$. $\Rightarrow \frac{f}{\delta} \in V$. $\Rightarrow |\varphi(\frac{f}{\delta})| < 1 \Rightarrow |\varphi(f)| < \delta$

$\varphi = \sum_{i=1}^{k} \lambda_i x_i \in E$.

$\varphi(f) = \langle f, x \rangle$, $\forall f \in E^*$, 对于某个 $x \in E$.

(Banach-Alaoglu) E 赋范，$(\overline{B_{E^*}}, \sigma(E^*,E))$ 紧。即

E^* 之闭单位球是弱*紧的。

Pf: $E^* \subset \mathbb{K}^E$. $\forall f \in E^*$. $\underline{\{f(x)\}_{x\in E}} \in \underline{\mathbb{K}^E}$.

$(E^*, \sigma(E^*,E))$ 可看成是 $\mathbb{K}^E$ 上之乘积拓扑.

(x_i^*) 是 E^* 中网列. $x_i^* \longrightarrow x^*$ 在弱*拓扑 $\iff x_i^* \longrightarrow x^*$ in $\mathbb{K}^E$.

$\iff \forall x \in E.\ \langle x_i^*, x\rangle \longrightarrow \langle x^*, x\rangle$.

$\iff x_i^* \longrightarrow x^*$ in $\mathbb{K}^E$.

$(E^*, \sigma(E^*,E))$ 可由 $\mathbb{K}^E$ 上之乘积拓扑诱导.

$\overline{B_{E^*}}$, $\sigma(E^*,E)|_{\overline{B_{E^*}}}$, 看成 $\mathbb{K}^E$ 在 $\overline{B_{E^*}}$ 上之限制

$\forall f \in E^*$. $f \longmapsto f|_{\overline{B_E}}$.

$\overline{B_{E^*}} = \{ x^* \in E^* \mid |x^*(x)| \le 1,\ \forall x \in \overline{B_E} \}$

则 $\overline{B_{E^*}}$ 中元素在此意义下取值于 $B_{\mathbb{K}} = K$.

$\forall f \in \overline{B_{E^*}}$. f 对应于 $f|_{\overline{B_E}} : \overline{B_E} \to K$. $\overline{B_{E^*}}$ 看成 $K^{\overline{B_E}}$ 之子集

K 紧. $K^{\overline{B_E}}$ 紧. $\overline{B_{E^*}}$ 闭 $\Rightarrow$ $\overline{B_{E^*}}$ 紧 (w-*紧)

$\forall \varphi \in \overline{B_{E^*}}$, φ 是 $\overline{B_E}$ 上之线性映射, $\forall x, y \in \overline{B_E}$, $\alpha, \beta \in \mathbb{K}$ s.t.

$\alpha x + \beta y \in \overline{B_E}$, $\varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y)$

若 $\psi : \overline{B_E} \to K$ 线性, 则 ψ 可延拓成 $\overline{B_{E^*}}$ 中之元.

$\forall x \in E$. 选择λ, s.t. $\frac{x}{\lambda} \in B_E$. $\varphi(x) := \lambda \psi(\frac{x}{\lambda})$

$\varphi \in \overline{B_{E^*}}$,

闭

设 $\psi \in K^{\overline{B_E}} \setminus \overline{B_{E^*}}$, 存在 $x_1, x_2 \in \overline{B_E}$, $\alpha, \beta \in \mathbb{K}$. $\alpha x_1 + \beta x_2 \in \overline{B_E}$

$\psi(\alpha x_1 + \beta x_2) \ne \alpha \psi(x_1) + \beta \psi(x_2)$.

记 $x_3 = \alpha x_1 + \beta x_2 \in B_Z$.

$V = V(\psi; x_1, x_2, x_3, \varepsilon) = \{ f \in K^{\overline{B_Z}} : |(f-\psi)(x_i)| < \varepsilon, \quad i=1,2,3 \}$

$|f(x_3) - \alpha f(x_1) - \beta f(x_2)| \geq |\psi(x_3) - \alpha\psi(x_1) - \beta\psi(x_2)|$ >0

$- |f-\psi(x_3)| - |\alpha| \, |f-\psi(x_1)| - |\beta| \, |f-\psi(x_2)|$

>0. 当ε足够小时.

$\Rightarrow V \subset K^{\overline{B_Z}} \setminus \overline{B_{Z^*}}$, $\Rightarrow$ $\overline{B_{Z^*}}$ 在 $K^{\overline{B_Z}}$ 闭 $\Rightarrow$ $\overline{B_{Z^*}}$ 紧. (吉洪诺夫)

$\Rightarrow$ $\overline{B_{Z^*}}$ 弱*紧.

(Goldstein). Z赋范. B_Z 在 $\overline{B_{Z^{**}}}$ 中弱*稠密.

$\overline{B_Z}^{w^*} = \overline{B_{Z^{**}}}$ $\left((Z^{**}, \sigma(Z^{**}, Z^*)) \right)$

pf: ① $B_Z \subset \overline{B_{Z^{**}}}$, $x \in B_Z$, $\hat{x} \in B_{Z^{**}}$, $x \longmapsto \hat{x}$.

从而 $\overline{B_Z}^{w^*} \subset \overline{B_{Z^{**}}}$ 只需说明 $\overline{B_{Z^{**}}}$ 是弱*闭的.

$\forall \|\varphi\|_{Z^{**}} > 1$. 找 φ的弱*邻域 V. s.t. $V \subset (\overline{B_{Z^{**}}})^c$.

$\exists f \in Z^*$. $\|f\|_{Z^*} \le 1$ $|\varphi(f)| > 1$.

$B(\varphi, \varepsilon) = \{\psi \in Z^{**} \mid |(\varphi-\psi)(f)| < \varepsilon\}$. φ的弱*邻域

ε足够小. $|\psi(f)| > 1$, $\Rightarrow$ $\psi \notin \overline{B_{Z^{**}}}$ $\Rightarrow$ $B(\varphi,\varepsilon) \subset (\overline{B_{Z^{**}}})^c$.

② $\overline{B_Z}^{w^*} = \overline{B_{Z^{**}}}$. 假设 $\exists \varphi \in \overline{B_{Z^{**}}} \setminus \overline{B_Z}^{w^*}$, $(Z^{**}, \sigma(Z^{**}, Z^*))$

$\{\varphi\}$ 单点集 紧. $\overline{B_Z}^{w^*}$ 闭

Z. (局部凸)

A, B 凸, A紧 B闭

$\exists f \in Z^*$. $f(A) < \alpha < f(B)$

根据 凸集分离定理 Thm. 存在

$f \in (Z^{**}, \sigma(Z^{**}, Z^*))^* = Z^*$. $\boxed{= Z^*}$

$f: (Z^{**}, \sigma(Z^{**}, Z^*)) \longrightarrow (K, \tau_K)$ 上的连续线性映射.

s.t. $\langle f,\varphi\rangle > 1$ 且 $\langle f, \overline{B_Z}^{w^*}\rangle \le 1$ $\qquad B_Z \subset \overline{B_Z}^{w^*}$

$\|f\|_{Z^*} = \sup_{x\in B_Z} |\langle f,x\rangle| = \sup_{x\in B_Z} |\langle \hat{x}, f\rangle| \le 1$

$\|\varphi\|_{Z^{**}} \ge |\langle f,\varphi\rangle| > 1$ 矛盾.

Z 赋范. $\tau: Z\to Z^{**}$, $x\mapsto \hat{x}$, 其中 $\hat{x}(f) = f(x)$, $\forall f\in Z^*$.

则 τ 是 Z 到 $\tau(Z)$ 上等距同构

$\|\hat{x}\|_{Z^{**}} = \|x\|_Z$

$\|x\|_Z = \sup_{\|f\|_{Z^*}\le 1} |\langle f,x\rangle| = \sup_{\|f\|_{Z^*}\le 1} |\langle \hat{x}, f\rangle| = \|\hat{x}\|_{Z^{**}}$.

τ 线性. $\qquad f(x_1-x_2)=0 \Rightarrow x_1-x_2=0$

τ 单? $\tau(x_1)=\tau(x_2)$, $\forall f\in Z^*$, $\underline{f(x_1)=f(x_2)} \Rightarrow x_1=x_2$.

$\forall y\ne 0$. $\exists f\in Z^*$. $f(y)\ne 0$

Def. Z 赋范. 称其自反. $\tau(Z)=Z^{**}$

eg. Hilbert 空间 H.

$H \cong H^* \cong H^{**}$

由 Riesz 定理. $\exists\ \Phi: H\to H^*$ 等距同构

$\Phi^*: H^{**}\to H^*$ 同构

$\underbrace{(\Phi^*)^{-1}\circ\Phi}_{=\tau}: H\to H^{**}$ 同构

$$\begin{tikzcd} H \arrow[rr, "(\Phi^*)^{-1}\circ\Phi"] \arrow[dr, "\Phi"'] & & H^{**} \\ & H^* \arrow[ur, "(\Phi^*)^{-1}"'] & \end{tikzcd}$$

$(\Omega,\mathcal{M},\mu)$

eg 2. (Ω, μ) σ有限测度空间. $1<p<\infty$ $L_p(\Omega,\mu)$ 自反.

$1<p<\infty$ $(L_p)^* \cong L_q$. $(L_p)^{**} \cong (L_q)^* \cong L_p$.

$\forall g \in L_q$ $\varphi_g(f) = \int_\Omega fg$. $\forall f \in L_p$ $\varphi_g \in (L_p)^*$.

$\|\varphi_g\| = \|g\|_{L^q}$ $L_q \subset (L_p)^*$.

反之 $(L_p)^* \subset L_q$ Radon-Nikodym.

$\forall \varphi \in (L_p)^*$ $Z \in \mu$. $\nu(Z) = \varphi(\mathbb{1}_Z)$ $\Rightarrow \nu$ 测度, $\nu \ll \mu$.

$\exists g \in L^1(\mu)$. s.t. $\nu(Z) = \int_Z \boxed{g}\, d\mu$.

h 简单函数 $h = \sum_{i=1}^n a_i \chi_{Z_i}$.

$\varphi(h) = \sum_{i=1}^n a_i \nu(Z_i) = \sum_{i=1}^n \int_{Z_i} a_i g\, d\mu = \int \sum_{i=1}^n a_i \chi_{Z_i} g\, d\mu$

$= \int hg\, d\mu$.

$\Rightarrow \varphi(f) = \int fg\, d\mu$. $\forall f \in L_p$

Z, F 赋范空间. $u \in \mathcal{B}(Z, F)$, 则 $\exists!\, u^* \in \mathcal{B}(F^*, Z^*)$ s.t.

对于 $\forall f^* \in F^*$, $x \in Z$, 有

$$\langle u^*(f^*), x \rangle = \langle f^*, u(x) \rangle.$$

特别地, u 同构, 则 u^* 同构.

Thm. Banach 空间 Z 自反 $\Leftrightarrow$ 它的对偶空间自反

Pf: "$\Rightarrow$" Z 自反 $Z = Z^{**}$ $\Rightarrow$ $Z^* = (Z^*)^{**}$.

$T: Z \to Z^{**}$. 等距同构

$T^*: Z^{***} \to Z^*$ 同构 $\Rightarrow$ $Z^* = (Z^*)^{**}$.

"$\Leftarrow$" Z^* 自反 $Z = Z^{**}$, 只需证明 Z 在 Z^{**} 中稠密 $\Rightarrow Z = Z^{**}$.

即证. 对于 $\forall f \in (Z^{**})^*$, $f|_Z = 0 \Rightarrow f = 0$

$f \in Z^{***}$, $\therefore$ $f \in Z^*$ $\quad f|_Z = 0 \Rightarrow f = 0$

(Ω, μ) σ-有限. $(L^1(\mu))^* = L^\infty(\mu)$

Thm. 对赋范空间 Z, F 同构. Z 自反 $\Leftrightarrow$ F 自反

证: F 自反 $\Rightarrow$ Z 自反

$u: Z \to F$ 同构, $\exists\, u^*: F^* \to Z^*$ 同构 $\exists\, u^{**}: Z^{**} \to F^{**}$ 同构

Claim. $u^{**}|_Z = u$ $\quad \forall x \in Z \quad u^{**}(\hat{x}) = \widehat{u(x)}$

$\forall f^* \in \underset{Z^{**}}{F^*}$. $\langle u^{**}\underset{Z^{**}}{(\hat{x})}, f^* \rangle = \langle \widehat{u(x)}, f^* \rangle$

$LHS = \langle \hat{x}, u^*(f^*) \rangle = \langle u^*(f^*), x \rangle = \langle f^*, u(x) \rangle$

$= \langle \widehat{u(x)}, f^* \rangle \Rightarrow u^{**}(\hat{x}) = \widehat{u(x)}$

F 自反 $F = F^{**}$. $\quad u^{**}|_Z = u : Z \to F$ 同构

$\qquad\qquad\qquad\qquad u^{**}: Z^{**} \to F^{**}$ 同构 $\Rightarrow Z = Z^{**}$

eg. L^1, L^∞ 不是自反的.

Z 自反, $\|x\| = \sup\limits_{\substack{f \in Z^* \\ \|f\|_{Z^*} \le 1}} |\langle f, x \rangle|$ 上确界可达, 即 $\|x\| = |\langle f, x \rangle|$

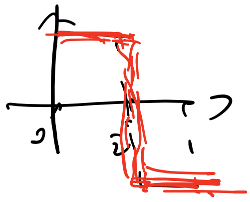
$\forall \varphi \in Z^*$. $\|\varphi\| = \sup\limits_{\substack{x^{**} \in Z^{**} \\ \|x^{**}\| \le 1}} |\langle x^{**}, \varphi \rangle| = \boxed{\sup\limits_{\substack{x \in Z \\ \|x\| \le 1}}} |\langle \varphi, x \rangle|$

$\exists\, x \in Z$, $\|x\| \le 1$. s.t. $\|\varphi\| = |\langle \varphi, x \rangle|$

eg1. $C[0,1]$ 不是自反的

$\varphi = \int_0^{\frac12} - \int_{\frac12}^1$ 线性、连续. $f_n \rightrightarrows f$. $\varphi(f_n)\to\varphi(f)$ $\Rightarrow$ $\varphi\in[C[0,1]]^*$

$\forall \|f\|_{L^\infty}\le 1$. $\sup\limits_{\|f\|_{L^\infty}\le 1} |\varphi(f)|$



$$|\varphi(f)| = \left|\int_0^{\frac12} f\,dx - \int_{\frac12}^1 f\,dx\right|$$

$$\le \int_0^{\frac12}\|f\|_{L^\infty}\,dx + \int_{\frac12}^1 \|f\|_{L^\infty}\,dx \le 1.$$

$\Rightarrow \|\varphi\|\le 1$

$\|\varphi\|\ge 1-\varepsilon$, $\forall\varepsilon>0$ $\Rightarrow$ $\|\varphi\|=1$.

$|\langle\varphi,f\rangle|=1$. $\|f\|_{L^\infty}=1$. a.e. $x\in[0,1]$ $f\equiv 1$. $f\equiv -1$

$|\langle\varphi,f\rangle|=0$. 矛盾.

eg2. $L^1[0,1]$ 不是自反的, $[L^1[0,1]]^* = L^\infty[0,1]$ $\Rightarrow$ $L^\infty[0,1]$ 不是自反的

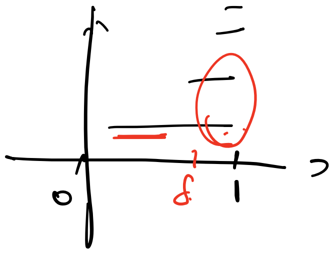
$\varphi(f) = \int_0^1 f(x)\,x\,dx$ φ 连续线性. $f_n\xrightarrow{L^1} f$ $\varphi(f_n)\to\varphi(f)$

$\forall \|f\|_{L^1}\le 1$. $|\varphi(f)|\le \|f\|_{L^1}\|x\|_{L^\infty} = \|f\|_{L^1}\le 1$. $\Rightarrow \|\varphi\|\le 1$.

$\|\varphi\|\ge 1-\varepsilon$. $f_n=\begin{cases} 0 & [0,1-\frac1n) \\ n & [1-\frac1n,1]\end{cases}$

$$\varphi(f_n) = \int_{1-\frac1n}^1 n x\,dx = \frac n2\left(1-(1-\tfrac1n)^2\right) = \frac12\left(2-\tfrac1n\right) = 1-\frac1{2n}$$

$\Rightarrow \|\varphi\|\ge 1-\varepsilon$ $\Rightarrow$ $\|\varphi\|=1$.

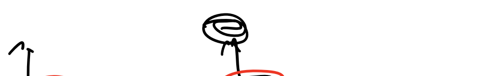


若 $\exists f\in L^1[0,1]$. $\|f\|_{L^1}\le 1$. $|\langle\varphi,f\rangle| = 1$.

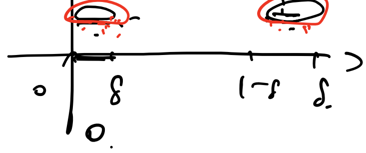
则 f 在 $[0,\delta]$ $=0$. a.e. $\forall\, 0<\delta<1$.

不妨假设 $0<\delta<\frac12$, 构造 $\tilde f$.

$\tilde f = \begin{cases} 0 & [0,\delta) \\ \ldots \end{cases}$



$$\hat f = \begin{cases} f & (\delta, 1-\delta) \\ f(x)+f(x-(1-\delta)) & (1-\delta, 1) \end{cases}$$



$\|\hat f\|_{L^1(0,1)} = \|f\|_{L^1(0,1)}$ 这的ĝ的φ的 $f\geq 0$

$$\int_0^1 |\hat f| = \int_\delta^{1-\delta} |f| + \int_{1-\delta}^1 f(x) + f(x-(1-\delta))$$

$$\qquad\qquad \overset{\shortparallel}{\int_0^\delta f(x)} \quad = \int_0^1 f(x) \leq 1$$

$\varphi(\hat f) > \varphi(f) = 1$ 矛盾. $\|\varphi\| \leq 1$

对一般的测度空间 $(X, \mathcal{M}, \mu)$. $1<p<\infty$ $\quad (L^p(\mu))^* = L^q(\mu)$

μ—有限的测度.

$p=1$. μ是σ-有限. $(L^1(\mu))^* = L^\infty(\mu)$

Thm. X对称. X自反 $\Leftrightarrow$ $\overline{B_X}$是w-紧的.

"$\Rightarrow$" X自反 $X = X^{**}$, $\overline{B_X} = \overline{B_{X^{**}}}$ $\quad \overline{B_{X^{**}}}$是弱*紧的 $\Rightarrow$ $\overline{B_X}$弱紧.

"$\Leftarrow$" $\overline{B_X}$是w-紧的. $\overline{B_X}$在$(X^{**}, \sigma(X^{**}, X^*))$弱*紧

$\Rightarrow$ $\overline{B_X}$在$(X^{**}, \sigma(X^{**}, X^*))$中闭(弱*)

$\Rightarrow$ $\overline{B_X} = \overline{B_X}^{w^*} \underset{\text{Goldstein}}{=} \overline{B_{X^{**}}}$ $\Rightarrow$ X自反

$\overline{B_{X^*}}$上三种拓扑. 强紧 $\Leftrightarrow$ $\dim X^* < \infty$ $\Leftrightarrow$ $\dim X < +\infty$

$\Downarrow$

弱紧 $\Leftrightarrow$ X自反

$\Updownarrow$

弱*紧