Serrin-type blow up criterion.

3D. $\|\rho(t)\|_{W^{1,q}} \le C$, $\|u(t)\|_{H^2} \le C$. — Choe-Kim 2004. local existence.

Serrin-type blow up: T^* maximal time of existence;

$$\lim_{t\to T^*} \|\rho\|_{L^\infty_T L^\infty} + \|u\|_{L^s_T L^q} = +\infty,$$

3D, classical. $\Rightarrow$ $7\mu > \lambda$. (viscosity assumption).

$$\Rightarrow \lim_{t\to T^*} \|\rho\|_{L^\infty_T L^\infty} = +\infty.$$

2D, Kazhikhov

$$\lim_{t\to T^*} \|\rho\|_{L^\infty_T L^\infty} = +\infty.$$

Sketch of the proof: $W^{1,q} \hookrightarrow C^\alpha$, strong solution,

$\|\rho\|_{L^\infty_T L^\infty} \le C$, $\Rightarrow$ $\|\rho\|_{W^{1,q}} + \|u\|_{H^2} \le C(\|\rho\|_{L^\infty_T L^\infty})$., $q > d = 3$.

hidden structures. $\longrightarrow$ $\|\nabla\rho\|_{L^q} \le C$,

Renormalization for $|\nabla\rho|^q$: $\to |\nabla\rho|^q$ $\qquad |\nabla\rho|^q \cdot |\nabla u|$.

$$\partial_t(|\nabla\rho|^q) + \nabla\cdot(|\nabla\rho|^q u) + (q-1)|\nabla\rho|^q \nabla\cdot u + q|\nabla\rho|^{q-2}\nabla\rho\cdot\nabla u\cdot\nabla\rho$$

$$+ q\rho|\nabla\rho|^{q-2}\nabla\rho\cdot\boxed{\nabla(\nabla\cdot u)} = 0.$$

$\rho|\nabla\rho|^{q-1} \cdot |\nabla^2 u|$ $\to$ high-order.

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C\left(\int |\nabla\rho|^q |\nabla u|\right)^{\frac{1}{q}} + C\left(\int \rho\, |\nabla\rho|^{q-1} |\nabla^2 u|\right)^{\frac{1}{q}}$$

$$\le C\|\nabla\rho\|_{L^q}\underbrace{\|\nabla u\|_{L^\infty}^{\frac{1}{q}}}_{} \sim 100 + C\underbrace{\|\nabla\rho\|_{L^q}^{\frac{q-1}{q}}\|\nabla^2 u\|_{L^q}^{\frac{1}{q}}}_{}$$

$$\le C\|\nabla\rho\|_{L^q}(\|\nabla u\|_{L^\infty}+1) + C(\|\nabla\rho\|_{L^q} + \|\nabla^2 u\|_{L^q})$$

$$\le C\|\nabla\rho\|_{L^q}(\underbrace{\|\nabla u\|_{L^\infty}}_{①}+1) + C\underbrace{\|\nabla^2 u\|_{L^q}}_{②} \rightarrow \underbrace{\|\nabla(\nabla\cdot u)\|_{L^q}}_{F+P} \rightarrow 2D$$

H^1

$Au = \rho\dot{u} + \nabla P$

$q \in (3,6) \leftarrow L^2 \cap L^6$

② $\|\nabla^2 u\|_{L^q} \le C(\|\rho\dot{u}\|_{L^q} + \|\nabla P\|_{L^q})$

$P = \rho^\gamma$ $\quad q=4$

$$\le C(\|\rho\dot{u}\|_{L^2\cap L^6} + \|\nabla\rho\|_{L^q})$$

$\rho \le C$

$$\le C(\|\rho\dot{u}\|_{L^2} + \|\nabla\dot{u}\|_{L^2} + \|\nabla\rho\|_{L^q})$$

Beale-Kato-Majda inequality

hidden structure

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C\|\nabla\rho\|_{L^q}(\underbrace{\|\nabla u\|_{L^\infty}}_{}+1) + \underbrace{C(\|\rho\dot{u}\|_{L^2} + \|\nabla\dot{u}\|_{L^2})}_{}$$

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① Beale-Kato-Majda inequality:

② hidden structure: $0 \to 1 \to 2 \longleftarrow \|\rho^{\frac{1}{2}}\dot{u}\|_{L^\infty_T L^2}^2 + \|\nabla\dot{u}\|_{L^2_T L^2}^2 \le C$

energy $\quad \|\nabla u\|_{L^\infty_T L^2} + \|\rho^{\frac{1}{2}} u_t\|_{L^2_T L^2}$

③ $\|\nabla^2 u\|_{L^\infty_T L^\infty} \leftarrow \|\rho\|_{W^{1,q}} \le C$ ⊕

① Beale–Majda–Kato inequality: 3D. d-

$$\|\nabla u\|_{L^\infty} \lesssim C(\|\nabla\cdot u\|_{L^\infty} + \|\nabla\times u\|_{L^\infty})\log(e+\|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C,\quad q>d=3.$$

$0+$

Proof: $\|\nabla u\|_{L^\infty} \leftarrow$ estimate

BKM 3D

$-\Delta u = f$ $\ln|x|$

$\Rightarrow u = P*f$, $P \to \frac{1}{4\pi|x|}$ (3d)

$$\Delta u = \nabla(\nabla\cdot u) - \nabla\times(\nabla\times u).$$

$$\Rightarrow u = \Delta^{-1}(\nabla(\nabla\cdot u)) - \Delta^{-1}(\nabla\times\nabla\times u).$$

$$= \frac{1}{4\pi}\int \frac{\nabla\times(\nabla\times u)(y)}{|x-y|} - \frac{1}{4\pi}\int \frac{\nabla(\nabla\cdot u)(y)}{|x-y|}$$

$$= \underbrace{\frac{1}{4\pi}\int \nabla_y \frac{1}{|x-y|}\times(\nabla\times u)(y)\,dy}_{w} + \boxed{\frac{1}{4\pi}\int \nabla_y \frac{1}{|x-y|}\,\nabla\cdot u(y)\,dy}_{v}.$$

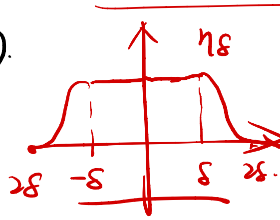
$\nabla u = \nabla w + \nabla v$.

$v = K*(\nabla\cdot u)$. $K = \frac{1}{|x|} \sim |x|^{-1}$

$K = \nabla\frac{1}{4\pi|x|} \sim \frac{1}{|x|^2}$

$$\nabla v = \nabla(K*(\nabla\cdot u)) = \nabla K*(\nabla\cdot u)$$

$$= \nabla K*(\nabla\cdot v) - \nabla(\eta_\delta K)*(\nabla\cdot u) + \eta_\delta K*(\nabla(\nabla\cdot u)).$$



$$= \nabla(1-\eta_\delta)K*(\nabla\cdot v) + \eta_\delta K*\nabla(\nabla\cdot v).$$

$$= -(\nabla\eta_\delta)K*(\nabla\cdot v) + (1-\eta_\delta)\nabla K*(\nabla\cdot v) + \eta_\delta K*\nabla(\nabla\cdot v)$$

$$= K_1*(\nabla\cdot v) + K_2*(\nabla\cdot v) + K_3*(\nabla(\nabla\cdot v)).$$

$\nabla K - \eta_\delta\nabla K$

$-\nabla\eta_\delta\cdot K$

$\mathrm{supp}\, K_1 = \{|x|\in[\delta,2\delta]\}$.

$\mathrm{supp}\, K_2 = \{|x|\in[\delta,+\infty)\}$

$\mathrm{supp}\, K_2 = \{|x|\in[0,2\delta]\}$

$$\|K_1*(\nabla\cdot v)\|_{L^\infty} \le \|\nabla\eta_\delta K\|_{L^1}\|\nabla\cdot v\|_{L^\infty}$$

$$\le C\int_\delta^{2\delta}(\delta^{-1}r^{-2})\,r^2\,dr\ \|\nabla\cdot v\|_{L^\infty}$$

r^{-1} r^1

$$\le C\|\nabla\cdot v\|_{L^\infty}$$

$\int_\delta^{2\delta}\delta^{-1}\,dr$. $\delta^{-1}\cdot\delta$. $\le C$

$$\|K_2 * (\nabla\cdot v)\|_{L^\infty} \le C\int_\delta^1 + \int_1^\infty |\nabla K(x-y)|\,|\nabla\cdot v(y)|\,dy$$

$$\le C\int_\delta^1 (r^{-3}) r^2\,dr \cdot \|\nabla\cdot v\|_{L^\infty} + C\int_1^\infty (r^{-3}) r^2\,dr\,\|\nabla\cdot v\|_{L^\infty}$$

$-\log\delta\,\|\nabla\cdot v\|_{L^\infty}$

$$C\left(\int_1^\infty (r^{-3})^{1+\varepsilon} r^2\,dr\right)^{\frac{1}{1+\varepsilon}} \|\nabla\cdot v\|_{L^{1+\frac{1}{\varepsilon}}}$$

$(\varepsilon = 1)$

$\frac{1}{1+\frac{1}{\varepsilon}}$

$\frac{\varepsilon}{\varepsilon+1} + \frac{1}{\varepsilon+1} = 1$

$$\le -C\log\delta\,\|\nabla\cdot u\|_{L^\infty} + C\varepsilon^{-\frac{1}{1+\varepsilon}}\|\nabla u\|_{L^{1+\frac{1}{\varepsilon}}}$$

$C\varepsilon^{-\frac{1}{1+\varepsilon}}\|\nabla\cdot u\|_{L^{1+\frac{1}{\varepsilon}}}$

$\|\nabla\cdot v\|_{L^2}$

$$\left(\int_1^\infty r^{-1-3\varepsilon}\right)^{\frac{1}{1+\varepsilon}}$$

$$\frac{1}{-3\varepsilon} r^{-3\varepsilon}\Big|_1^\infty$$

$$\left(0 - \frac{1}{-3\varepsilon}\right)^{\frac{1}{1+\varepsilon}}$$

$r^{-1-2\varepsilon}$

$$\|K_3 * (\nabla\cdot v)\|_{L^\infty} \le \|\eta_\delta K\|_{L^{\frac{q}{q-1}}} \|\nabla^2 v\|_{L^q}$$

$$\le C\left(\int_0^\delta (r^{-2})^{\frac{q}{q-1}} r^2\right)^{\frac{q-1}{q}} \|\nabla^2 v\|_{L^q}$$

r^{1-d}, $d-1$, $\frac{q}{q-1}$, $\frac{(d-1)q}{q-1}$, $q > 2 = d$, $\frac{d-1}{q-1}$

$$\left(\delta^{\frac{q-3}{q-1}}\right)^{\frac{q-1}{q}}$$

$\delta^{\frac{q-3}{q}}$

$\delta^{\frac{q-d}{q}}$

$$\le C\delta^{\frac{q-d}{q}} \|\nabla^2 v\|_{L^q}$$

$$\delta = \min\left\{1, \|\nabla^2 u\|_{L^q}^{-\frac{q}{q-d}}\right\}$$

$$\int_0^\delta r^{-\frac{2q}{q-1}}$$

$$\|\nabla v\|_{L^\infty} \le C\delta^{\frac{q-d}{q}}\|\nabla^2 u\|_{L^q} + C(1-\log\delta)\|\nabla\cdot u\|_{L^\infty} + C\|\nabla u\|_{L^2}$$

$1 - \frac{2}{q-1} = \frac{q-3}{q-1}$, $(q>3)$

$1 - \frac{d-1}{q-1} = \frac{q-d}{q-1}$

$\log(e + \|\nabla^2 u\|_{L^q})$

$$\le C + C(q)\left(1 + \max\{0, \log\|\nabla^2 u\|_{L^q}\}\right)\|\nabla\cdot u\|_{L^\infty} + C\|\nabla u\|_{L^2}$$

$$\frac{q-1}{q-3} r^{\frac{q-3}{q-1}}\Big|_0^\delta$$

$$\le C + C(q)\|\nabla\cdot u\|_{L^\infty}\log(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2}$$

$\frac{q-d}{q}$

$\|\nabla w\|_{L^\infty} \le$

$\|\nabla\times w\|_{L^\infty}\log(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla w\|_{L^2}$

$\delta^{\frac{q-3}{q-1}}$

Application of BKM inequality:

$\int_0^t \|\nabla \dot u\|_{L^2}^2 \leftarrow$ second hidden structure

$$\frac{d}{dt}\|\nabla \rho\|_{L^q} \le C\|\nabla \rho\|_{L^q}\left(\|\nabla u\|_{L^\infty}+1\right) + C\|\nabla \dot u\|_{L^2} + C$$

$q > d$

$C \to$ first

$$\|\nabla u\|_{L^\infty} \le C\left(\|\nabla\cdot u\|_{L^\infty}+\|w\|_{L^\infty}\right)\log\left(e+\|\nabla^2 u\|_{L^q}\right) + C\|\nabla u\|_{L^2} + C$$

$$\le C\left(\|\nabla\cdot u\|_{L^\infty}+\|w\|_{L^\infty}\right)\left(\log\left(e+\|\nabla \dot u\|_{L^2}\right)+\log\left(e+\|\nabla \rho\|_{L^q}\right)\right) + C$$

$$\frac{d}{dt}\|\nabla \rho\|_{L^q} \le C\left(\|\nabla\cdot u\|_{L^\infty}+\|w\|_{L^\infty}\right)\left(\log\left(e+\|\nabla \rho\|_{L^q}\right)\right)\|\nabla \rho\|_{L^q} \le f$$

$\le g \qquad \le \ln f$

$e+ \qquad \le g \qquad + C\left(\|\nabla\cdot u\|_{L^\infty}+\|w\|_{L^\infty}\right)\log\left(e+\|\nabla \dot u\|_{L^2}\right)\|\nabla \rho\|_{L^q} \le f$

$\le g$

$$+ C\|\nabla \rho\|_{L^q} + C\|\nabla \dot u\|_{L^2}^2$$

$\le g$

$\le g \qquad \le f$

$$f = e + \|\nabla \rho\|_{L^q} \qquad g(t) = C\left(1+\|\nabla\cdot u\|_{L^\infty}+\|w\|_{L^\infty}+\|\nabla \dot u\|_{L^2}\right)\log\left(e+\|\nabla \dot u\|_{L^2}\right)$$

$g \in L^1$

$\left(\|\nabla\cdot u\|_{L^\infty}+\|w\|_{L^\infty}\right)^1$

$\|\nabla \dot u\|_{L^2}^{1+}$

$\|\nabla \dot u\|^{0+}$

$\|\nabla \dot u\|_{L^1}^{1+}$

$$\frac{d}{dt} f \le C g \ln f\, f + C g f$$

$g \in L^1.$

$$\frac{d}{dt}\ln f \le C g \ln f + C g \le C g\left(\ln f + 1\right)$$

L^2

Gronwall inequality

$$\ln f(t) \le C(T)$$

$$\downarrow$$

$$f(t) \le C(T)$$

$$\downarrow$$

$$\|\nabla \rho\|_{L^q} \lesssim C(T) \rightarrow$$

Euler — NLS

NS.

renormalization

$u = e^{i\phi} w$

$w \to u$, ρ

Euler C^∞ self-similar $\longrightarrow$ NLS

NS Δ $\to$ linearization

$\partial_t u + \Delta u$

implosion

shock

blow-up $\to$ $\nabla u \to \infty$

$\boxed{u \to \infty}$

implosion

Fluid: $d = 2$ $d \geq 3$

NLS: $d = 4$, $d - 2\sqrt{d} = 0$, $d \geq 5$