View previous:

① (Boundary estimates), $\partial\Omega \in C^{2,\alpha}$ $(0<\alpha<1)$.

$u \in C^{2,\alpha}(\bar\Omega)$ satisfies $\begin{cases} -a^{ij}D_{ij}u + b^i D_i u + cu = f & \text{in } \Omega \\ u = \varphi \ (0) & \text{on } \partial\Omega \end{cases}$

① Uniformly elliptic: $\exists \Lambda \geq \lambda > 0$ ①

s.t. $\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$, $\forall x \in \Omega$, $\xi \in \mathbb{R}^n$.

② $a^{ij}, b^i, c \in C^\alpha(\bar\Omega)$ $(0<\alpha<1)$ and ②

$\frac{1}{\lambda}\left(\sum_{i,j}|a^{ij}|_{\alpha;\Omega} + \sum_i |b^i|_{\alpha;\Omega} + |c|_{\alpha;\Omega}\right) \leq \Lambda_\alpha$

$\Rightarrow |u|_{2,\alpha;\Omega} \leq C\left[\frac{1}{\lambda}|f|_{\alpha;\Omega} + \boxed{|u|_{0;\Omega}}\right]$ $(\varphi = 0)$.

$\tilde u = u - \varphi$

$|u|_{2,\alpha;\Omega} \leq C\left[\frac{1}{\lambda}|f|_{\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega} + |u|_0\right]$

② $\boxed{c \geq 0}$, $f \in C^\alpha(\bar\Omega)$, $\varphi \in C^{2,\alpha}(\bar\Omega)$ $(0<\alpha<1)$ ③

$u \in C^2(\Omega) \cap C(\bar\Omega)$.

$|u|_{0;\Omega} \leq \sup_{\partial\Omega}|u| + C|f|_{0;\Omega}$ φ, f.

§7. Dirichlet problems

$$\begin{cases} -a^{ij}D_{ij}u + b^i D_i u + cu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \quad ④$$

$\boxed{C^{2,\alpha}}$

Lemma 7.1. let $\partial\Omega \in C^{[\frac{n}{2}]+4}$ (C^∞). The coefficients satisfy ①②③.

$\Rightarrow$ ④ $\exists!$ $u \in C^{2,\alpha}(\bar{\Omega})$

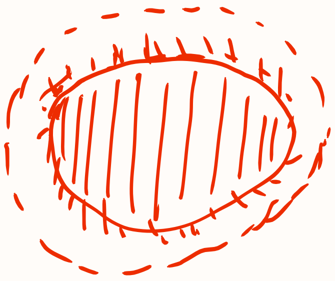
proof: Without loss of generality $\varphi \equiv 0$ $(\tilde{u} = u - \varphi)$

L^2 Theory $\longrightarrow$ Schauder Theory : $f_N \rightrightarrows f$ C^α.

$f \in C^\alpha$. $\varphi \in C^{2,\alpha}$.

Let a_N^{ij}, b_N^i, c_N, $f_N \in C^\infty(\bar{\Omega})$. s.t

① $a_N^{ij} \rightrightarrows a^{ij}$, $b_N^i \rightrightarrows b^i$, $c_N \rightrightarrows c$, $f_N \rightrightarrows f$ $(i,j = 1,2,\cdots,n)$



$f \in C$.

$a_N^{ij}\xi_i\xi_j \rightrightarrows a^{ij}\xi_i\xi_j$ ✓

$N \to +\infty$

② $\frac{\lambda}{2}|\xi|^2 \le a_N^{ij}\xi_i\xi_j \le 2\Lambda|\xi|^2$, $\forall x \in \Omega$, $\xi \in \mathbb{R}^n$ ✓ (N sufficiently large)

③ $c_N \ge 0$ ✓; $\|f_N\|_{\alpha;\Omega} \le 2\|f\|_{\alpha;\Omega}$ $\quad C^{\alpha'}$ $(\alpha' < \alpha)$

$\frac{1}{\lambda}\left\{\sum_{i,j}|a_N^{ij}|_\alpha + \sum_i |b_N^i|_\alpha + |c_N|_\alpha\right\} \le 2\Lambda_\alpha$.

$f_N \rightrightarrows f$ $\quad C^{0,\alpha'}$ $\quad \alpha' \to \alpha$

$\|f_N\|_{\alpha';\Omega} \le 2\|f\|_{\alpha';\Omega}$. $\quad \alpha' \to \alpha$.

Approximation Problem:

$$\begin{cases} -a_N^{ij} D_{ij} u_N + b_N^i D_i u_N + c_N u_N = f_N, & \text{in } \Omega \\ u_N = 0. & \text{on } \partial\Omega \end{cases} \quad C^\infty$$

(Uniqueness Theory)

$\boxed{L^2\text{-Theory}}$: Fredholm Alternative Theory

$$\begin{cases} Lu = f \\ u = 0 \end{cases} \quad \text{and} \quad \begin{cases} Lu = 0 \\ u = 0 \end{cases}$$

det$A \neq 0$ Uniqueness or $u \neq 0$ det $A = 0$

$C_N \geq 0$ $|u|_{C^{2,\alpha}} \leq |f|_{0,\alpha} + |\varphi|_{2,\alpha} = 0$ $u \equiv 0$

$u \in H_0^1(\Omega)$ $\exists!$

$\partial\Omega \in C^{[\frac{n}{2}]+4}$ C^∞ $\partial\Omega \in C^{[\frac{n}{2}]+4}$ $a^{ij} \in H^{[\frac{n}{2}]+4}$

$\downarrow$

Regularity Theory: $u_N \in H^{[\frac{n}{2}]+4}(\Omega) \cap H_0^1(\Omega)$ $u_N \in H^{[\frac{n}{2}]+4}$

$\partial\Omega \in C^\infty \Rightarrow u_N \in C^\infty(\Omega)$. a version

$\partial\Omega \in C^{[\frac{n}{2}]+4}$ $u_N \in H^{[\frac{n}{2}]+4}(\Omega)$.

Sobolev-embedding Thm.

$u_N \in C^{2,\alpha}(\bar{\Omega})$. $C^{k-[\frac{n}{p}]-1,\gamma}$

C^α $k - \frac{n}{p} \geq 3$. $[\frac{n}{2}]+4-[\frac{n}{2}]-1 = 3$

$\gamma = \frac{1}{2}$, $\forall$ Cn

$k - \frac{n}{p} \notin \mathbb{Z}$ $C^{k-\frac{n}{p}} = C^{[\frac{n}{2}]+4-\frac{n}{2}} \geq 3$

$u_N \in C^{2,\alpha}(\bar{\Omega})$. Schauder Boundary estimate:

$$|u_N|_{2,\alpha;\Omega} \leq C\|f_N\|_{\alpha;\Omega} \leq 2C\|f\|_{\alpha;\Omega}$$

By Ascoli-Arzelà Thm.

$$\Rightarrow \{u_N\} \ni \{u_{N_k}\} \rightrightarrows u \quad (C^2(\bar{\Omega}))$$

$$\Uparrow$$

$$\begin{cases} D^2 u_N \rightrightarrows D^2 u \\ D u_N \rightrightarrows D u \\ u_N \rightrightarrows u \end{cases} \quad (\text{Closed operator})$$

$$\begin{cases} -a_N^{ij} D_{ij} u_N + b_N^i D_i u_N + c_N u_N = f_N & \text{in } \Omega \\ u_N = 0, & \text{on } \partial\Omega \end{cases}$$

$N \to +\infty$

$f \in C^{0,\alpha}$

$$\begin{cases} -a^{ij} D_{ij} u + b^i D_i u + c u = f & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

$\to u \in C^2(\bar{\Omega})$

$u_N \rightrightarrows u \ (C^2)$

$[D_{ij} u_N]_\alpha$

$\Downarrow$

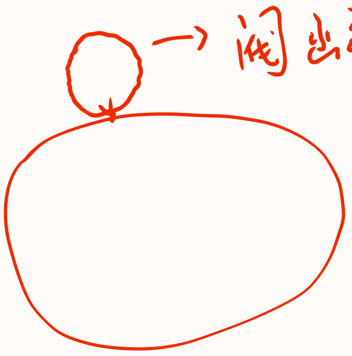
$[D_{ij} u]_\alpha < +\infty$

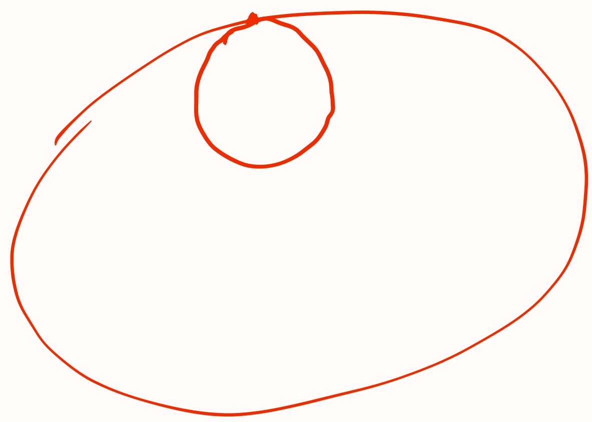
$D_{ij} u \in C^\alpha$

$\Downarrow$

$\Rightarrow u \in C^{2,\alpha}(\bar{\Omega})$

→ 闸函数

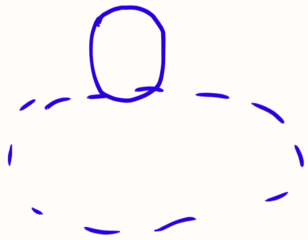


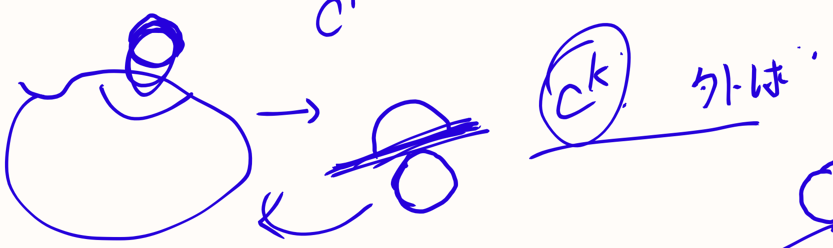


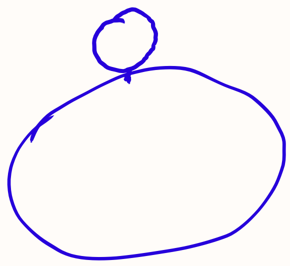
Def 1. We say a bounded domain $\Omega \subseteq \mathbb{R}^n$ has outside ball proposition:

If for $\forall x_0 \in \partial\Omega$ $\exists B_R(y) \subset \mathbb{R}^n \setminus \bar{\Omega}$ s.t

$$B_R(y) \cap \bar{\Omega} = \{x_0\}.$$







Thm 2. let Ω has outside ball proposition

① ② . $c \geq 0$. $f \in C^{\alpha}(\bar{\Omega})$. $\varphi \in C(\bar{\Omega})$. Dirichlet Problem has

$\exists!\ u \in \cancel{C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})}$ $C^{2,\alpha}(\bar{\Omega})$. $\Omega_N \in C^{[\frac{n}{2}]+4}$

proof: let $\{\Omega_N\}$, $\Omega_N \subset \Omega$. $\Omega_1 \subset \Omega_2 \dots \subset \Omega_N \subset \dots$

$\partial\Omega_N \in C^{[\frac{n}{2}]+4}$. $\text{dist}\{\bar{\Omega}_N, \partial\Omega\} \leq \frac{1}{N}$.

$\varphi \in C(\bar{\Omega})$. $\{\varphi_N\}$. $\varphi_N \in C^{2,\alpha}(\bar{\Omega})$, $|\varphi_N - \varphi|_{0;\Omega} \leq \frac{1}{N}$.

Consider Dirichlet Problem:

$$\begin{cases} -a^{ij} D_{ij} u_N + b^i D_i u_N + c u_N = f & \text{in } \Omega_N \\ u_N = \varphi_N & \text{on } \partial\Omega_N \end{cases} \quad ⑤$$

By lemma 7.1. $\exists!\ u_N \in C^{2,\alpha}(\bar{\Omega}_N)$ satisfies ⑤.

For any $\Omega' \subset\subset \Omega$ by Schauder interior estimates:

(N sufficiently large, $\Omega' \subset\subset \Omega_N$)

$$|u_N|_{2,\alpha;\Omega'} \le C\{|f|_{\alpha,\Omega} + |u_N|_{0;\Omega_N}\}$$

$$\le C\{|f|_{\alpha;\Omega} + |\varphi|_{0;\Omega} + \frac{1}{N}\}.$$

By Ascoli - Arzelà Thm.

$\{u_N\}$ has a convergence subsequence in $C^2(\bar{\Omega}')$.

$$C^{0,\frac{1}{2}} \quad C^{0,\frac{1}{3}} \quad C^{0,\frac{1}{2}} \subset\subset C^{0,\frac{1}{3}}$$

Ω_1, $\{u_N\} \to \{u_{N_{k_1}}\} \rightrightarrows u$ $(C^2(\bar{\Omega}_1))$

Ω_2, $\{u_{N_{k_2}}\} \rightrightarrows u$ $(C^2(\bar{\Omega}_2))$

$\vdots$

Ω_n, $\{u_{N_{k_n}}\} \rightrightarrows u$ $(C^2(\bar{\Omega}_n))$.

$\vdots$

$u \in C^{2,\alpha}(\Omega)$.

$\{u_{N_k}\}$: $\exists\, u \in C^{2,\alpha}(\Omega)$ $\forall \Omega' \subset\subset \Omega$. $k \to \infty$ $u_{N_k} \rightrightarrows u\, (C^2(\bar{\Omega}'))$

$\Rightarrow -a^{ij}D_{ij}u + b^i D_i u + cu = f$ ✓.

Only to prove:

u ① $C(\bar{\Omega})$

② $u = \varphi$, on $\partial\Omega$

$u \in C(\bar{\Omega})$. $u = \varphi$. on $\partial\Omega$

Using the barrier function method:

For $x_0 \in \partial\Omega$ let $B_\rho(y)$ be the exterior sphere as described in Def 2.1

The barrier function at x_0 is a function $w(x)$:

① $w(x_0)=0$, $w(x)>0$ for $x \in \bar{\Omega}\setminus\{x_0\}$

② $w \in C^2(\bar{\Omega})$. $Lw > 0$

$|x_0-y|=\rho$

$$w(x) = e^{-\beta\rho^2} - e^{-\beta|x-y|^2}, \quad \beta \text{ is to be determined}$$

$Lw \geq \theta > 0$

$w(x_0) = 0$ ✓ $\quad w(x)>0$. $x \in \bar{\Omega}\setminus\{x_0\}$

$-D_{ij}w \quad D_{ij}w < 0$

$$Lw \geq [4\lambda\beta^2\rho^2 - 2n\Lambda\beta - 2\beta\lambda\Lambda_\alpha\rho]e^{-\beta(d+\rho)^2} \geq \theta > 0$$

(β sufficiently large)

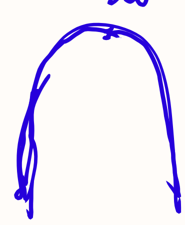
$d = \operatorname{diam}\Omega$

$\cup \quad \cap$

For any $\varepsilon > 0$, there exists a neighborhood $N(x_0)$ of x_0 s.t.

$|\varphi(x)-\varphi(x_0)| < \varepsilon$ for $x \in N(x_0)\cap\Omega$.

$\Omega \setminus N(x_0)$. Since $w(x)$ is bounded from below by a positive constant. $\theta'' \geq w(x) \geq \theta' > 0$



We can choose C large enough (depending on ε) s.t.

$$\boxed{Cw(x) + \varphi(x_0) + \varepsilon > \varphi(x) > -Cw(x) + \varphi(x_0) - \varepsilon, \quad x \in \bar{\Omega}}$$ ✓

$x \in \Omega \cap N(x_0)$ ✓ $\quad |\varphi|_0 < +\infty \quad \varphi \in C(\bar{\Omega})$.

$x \in \Omega \setminus N(x_0)$.

$|\varphi_N - \varphi|_{0;\Omega} \le \frac{1}{N}$.

If N is sufficiently large

① $\Rightarrow$ $Cw(x)+\varphi(x_0)+\varepsilon \geq \varphi_N(x) \geq -Cw(x)+\varphi(x_0)-\varepsilon$, $x \in \bar{\Omega}$

$\because$ $Lw \geq \theta > 0$.

$\Rightarrow$ For sufficently large C.

② $\begin{cases} L(Cw(x)+\varphi(x_0)+\varepsilon) \geq Lu_N \geq L(-Cw(x)+\varphi(x_0)-\varepsilon), & x \in \bar{\Omega}_N \\ -Cw(x)+\varphi(x_0)-\varepsilon \le \varphi_N = u_N \le Cw(x)+\varphi(x_0)+\varepsilon, & x \in \partial\Omega_N \end{cases}$

Weak Maximum Principle:

$\Rightarrow$ $Cw(x)+\varphi(x_0)+\varepsilon \geq u_N \geq -Cw(x)+\varphi(x_0)-\varepsilon$, $x \in \Omega_N$.

$N \to \infty$. $\Rightarrow$ $Cw(x)+\varphi(x_0)+\varepsilon \geq u(x) \geq -Cw(x)+\varphi(x_0)-\varepsilon$, $x \in \Omega$.

$\downarrow$ $w(x_0)=0$.

letting $x \to x_0$,

$\Rightarrow$ $\varphi(x_0)+\varepsilon \geq \limsup_{x \to x_0} u(x) \geq \liminf_{x \to x_0} u(x) \geq \varphi(x_0)-\varepsilon$.

$\because \forall \varepsilon$

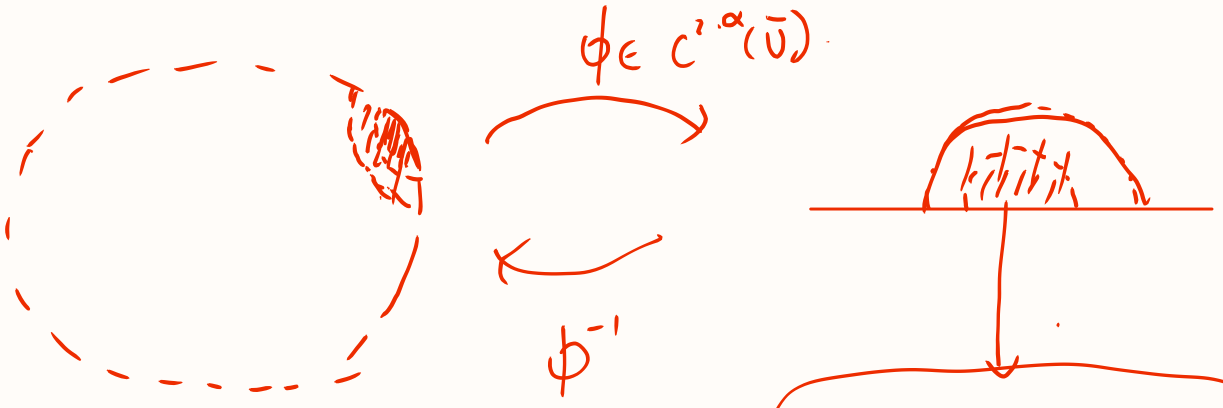
$\Rightarrow$ $\lim_{x \to x_0} u(x) = \varphi(x_0)$.

Thm 7.3 let $\partial\Omega \in C^{2,\alpha}$. $c \le 0$ $f \in C^{\alpha}(\bar{\Omega})$. $\varphi \in C^{2,\alpha}(\bar{\Omega})$

$\Rightarrow \exists!\ u \in C^{2,\alpha}(\bar{\Omega})$ Thm 7.2.

$\Rightarrow$ $u \in C^{2,\alpha} \cap C(\bar{\Omega})$ $\longrightarrow$ $u \in C^{2,\alpha}(\bar{\Omega})$

proof:



$u \in C^{2,\alpha}(B_1^+) \cap C(\overline{B_1^+})$

$\Rightarrow u \in C^{2,\alpha}(\overline{B^+_{\frac{1}{2}}})$

$\varphi \in C(\bar{\Omega})$

$$\begin{cases} Lu = f & \text{in } B_1^+ \\ u = \varphi & \text{on } \partial B_1^+ \end{cases}$$

$\partial B_1^+ \to$ sufficiently smooth.

Take $\varphi_N \in C^{2,\alpha}(\overline{B_1^+})$ s.t. $\varphi_N \big|_{l_1} = 0$

and $|\varphi_N - \varphi|_{0;B_1^+} \to 0 \quad (N \to +\infty)$.

Consider boundary value problem:

$$\begin{cases} -a^{ij} D_{ij} u_N + b^i D_i u_N + c u_N = f & \text{in } B_1^+ \\ u_N = \varphi_N, \ \text{on } \partial B_1^+ & \varphi_N \rightrightarrows \varphi. \end{cases}$$

$u = \varphi$

has a solution $u_N \in C^{2,\alpha}(\overline{B_1^+})$ by Thm 1.

$u_N \rightrightarrows \tilde{u} = u$

$$\begin{cases} u_{N_k} \rightrightarrows \tilde{u} \quad (C^2(\Omega')) \quad (\forall\, \Omega' \subset\subset B_1^+). \\ u_{N_k} \rightrightarrows \hat{u} \quad C(\bar{\Omega})), \end{cases}$$

$N_K \to \infty$

$$\begin{cases} -a^{ij} D_{ij} \hat{u} + b^i D_i \hat{u} + c\hat{u} = f & \text{in } B_1^+ \\ \hat{u} = u & \text{on } B_1^+ \end{cases}$$

$\hat{u} - u = u'$.

$\Rightarrow \underline{u = \hat{u}}$ (Maximum Principle).

$$\because \underbrace{[D^2 u_N]_{\alpha; B^+_{\frac{1}{2}}} \le C\left\{ |f|_{\alpha; B_1^+} + |u_N|_{0; B_1^+} \right\}}$$

$N \to +\infty$

$[D^2 \hat{u}]_{\alpha; B^+_{\frac{1}{2}}} \le \sim$

$\hat{u} \in C^{2,\alpha}(\overline{B^+_{\frac{1}{2}}}) \Rightarrow \underline{u \in C^{2,\alpha}(\overline{B^+_{\frac{1}{2}}})}$ ✓