

The Hahn-Banach Theorem

§ 1.1. 线性泛函

Def. 1. (次线性泛函). $p: E \rightarrow \mathbb{R}$ 满足.

i) $p(\lambda x) = \lambda p(x), \quad \forall x \in E, \lambda > 0$

ii) $p(x+y) \leq p(x) + p(y), \quad \forall x, y \in E.$

Thm 1. E 是实向量空间, $F \subset E$ 子空间, 余维数为 1. $p: E \rightarrow \mathbb{R}$ 次线性泛函.

f 是 F 上的线性泛函, $f|_F$ 满足 $f(x) \leq p(x), \quad \forall x \in F.$

则 $\exists \tilde{f}: E \rightarrow \mathbb{R}$ 线性泛函, s.t. $\tilde{f}|_F = f, \quad \tilde{f}(x) \leq p(x), \quad \forall x \in E.$

Pf. 任取 $x_0 \in E \setminus F, \quad E = F + \mathbb{R}\{x_0\}$

$|\tilde{f}(x)| \leq |p(x)|$

定义 $\tilde{f}(tx_0 + x) = t\tilde{f}(x_0) + \tilde{f}(x) = ta + f(x), \quad \forall t \in \mathbb{R}, x \in F.$

$\tilde{f} \leq p \Leftrightarrow ta + f(x) \leq p(tx_0 + x), \quad \forall t \in \mathbb{R}, x \in F.$
 $\Leftrightarrow ta \leq p(tx_0 + x) - f(x)$
 $\forall t \in \mathbb{R}, x \in F$ 成立.

$t > 0 \quad a \leq \frac{1}{t} p(tx_0 + x) - \frac{1}{t} f(x)$

$t < 0 \quad a \geq \frac{1}{t} p(tx_0 + x) - \frac{1}{t} f(x)$

$$\sup_{\substack{S < 0 \\ y \in F}} \frac{1}{S} p(Sx_0 + y) \leq (a) \leq \inf_{\substack{t > 0 \\ x \in F}} \frac{1}{t} p(tx_0 + x) - \frac{1}{t} f(x)$$

$\forall S < 0, t > 0, x, y \in F. \quad \frac{1}{S} p(Sx_0 + y) - \frac{1}{S} f(y) \leq \frac{1}{t} p(tx_0 + x) - \frac{1}{t} f(x)$

$\Leftrightarrow \frac{1}{t} p(tx_0 + x) - \frac{1}{S} p(Sx_0 + y) \geq \frac{1}{S} f(y) - \frac{1}{t} f(x)$

$\Leftrightarrow p(x_0 + \frac{x}{t} - x_0 - \frac{y}{S}) = p(\frac{x}{t} - \frac{y}{S}) \geq f(\frac{x}{t} - \frac{y}{S})$
 $\frac{x}{t} - \frac{y}{S} \in F, \quad \#$

Thm 2. (H-B 延) $E, F \subset E$ 子空间, f 是 F 上的线性泛函, p 如上.

$f: F \rightarrow \mathbb{R}$ 线性, $f \leq p.$

则存在 $\tilde{f}: E \rightarrow \mathbb{R}$ 线性, 且 $\tilde{f} \leq p.$

Pf. f 由 (G, g) 元组构成. (G, g) 满足

(1). $F \subset G \subset \mathbb{C}$, G 是 \mathbb{C} -子空间

(2). g 线性. g 是 G 上的线性泛函. $g|_F = f$.

定义 F 上一族泛函 \leq

$$(G, g) \leq (H, h) \iff G \subset H, h|_G = g.$$

则 F 上任一族泛函有上界. $G \subset F$. 令 F .

$$\text{令 } H = \bigcup_{G \in \mathcal{G}} G, \quad h: H \rightarrow \mathbb{R}, \text{ 若 } x \in G, \text{ 定义 } h(x) = g(x)$$

$(G, g) \leq (H, h)$, 由 Zorn's 引理, F 有极大元 (M, m)

则 $M = \mathbb{C}$. 若不然, $\exists x_0 \in \mathbb{C} \setminus M$. 考虑 $M \cup \mathbb{R}\{x_0\}$, $m \rightarrow \tilde{m}$

$$(M \cup \mathbb{R}\{x_0\}, \tilde{m}) \geq (M, m). \text{ 矛盾. } \#$$

Thm. (H-B 复). \mathbb{C} - F 上, 复. $p: \mathbb{C} \rightarrow \mathbb{R}$. 半范. $f: F \rightarrow \mathbb{C}$. 线性且 $|f| \leq p$

则存在 $\tilde{f}: \mathbb{C} \rightarrow \mathbb{C}$ 线性. $|\tilde{f}| \leq p$.

Remark. \mathbb{C} 上复线性泛函和实线性泛函同构

$$f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x) \quad \operatorname{Re} f(x) \text{ 记为 } \varphi(x)$$

$$f(ix) = i \operatorname{Re} f(x) - \operatorname{Im} f(x).$$

$$\text{则 } \operatorname{Im} f(x) = \varphi(ix) \Rightarrow \operatorname{Im} f(x) = -\varphi(ix)$$

$$f = \varphi(x) - i\varphi(ix)$$

$$\underline{f \text{ 复 } \varphi = \operatorname{Re} f. \mathbb{C} \text{ 上 } \varphi \rightarrow \tilde{\varphi}.$$

$$\tilde{f} = \tilde{\varphi}(x) - i\tilde{\varphi}(ix)$$

Cor. 1. \mathbb{C} - F 上. $g: F \rightarrow \mathbb{R}$. 实线性泛函. 则 $\exists f: \mathbb{C} \rightarrow \mathbb{C}$ 复线性泛函,

$$\text{且 } \|f\|_{\mathbb{C}^*} = \|g\|_{F^*}.$$

pf: 考虑 F . $|g(x)| \leq \|g\|_{F^*} \|x\|$ $\|g\|_{F^*} \|\cdot\|$

由 H-B 定理, $\exists f \in E^*$ s.t.

$$|f(x)| \leq \|g\|_{F^*} \|x\| \Rightarrow f \in E^*. \text{ 且 } \|f\|_{\mathbb{C}^*} \leq \|g\|_{F^*}$$

$$\text{又 } f|_F = g. \quad \|f\|_{\mathbb{C}^*} \geq \|g\|_{F^*} \Rightarrow \|f\|_{\mathbb{C}^*} = \|g\|_{F^*}$$

Cor 2. Z 同上, $\forall x \in Z, x \neq 0, \exists f \in Z^*$. s.t. $f(x) = \|x\|$, 且 $\|f\| = 1$

Pf: 考虑 $(\mathbb{R}^1)^*$. 定义 $g(x) = t\|x\|$. g 线性, 连续. $\|g\| \leq 1, \Rightarrow \|g\| = 1$.

由 Cor 1. $\exists f \in Z^*$. s.t. $\|f\|_{Z^*} = 1$. 且 $f(x) = g(x) = \|x\|$. \forall

eg. $\sigma(Z, Z^*)$ 是 Hausdorff 的.

$Z^* = \{f_i\}_{i \in I}$. $\{f_i\}_{i \in I}$ 是 Z 上的一族线性泛函, 定义 "开球"

选取 $J \subset I$. 有限子集,

$$B_J(x, \delta) = \{y \in Z \mid \max_{i \in J} |f_i(x-y)| < \delta\}$$

开球作为基生成拓扑即为 $\sigma(Z, Z^*)$

由于 Cor 1, $\forall x \neq y, \exists f \in Z^*$. s.t. $f(x) \neq f(y)$. $\Rightarrow \sigma(Z, Z^*)$ 是 Hausdorff.

eg. $\tau: Z \rightarrow Z^{**}, x \mapsto x^{**}, \|x\|_Z = \|x^{**}\|_{Z^{**}}$.

$$\|x^{**}\|_{Z^{**}} = \sup_{\|f\|_{Z^*} \leq 1} \langle x^{**}, f \rangle$$

$$= \sup_{\|f\|_{Z^*} \leq 1} \langle f, x \rangle$$

$$\rightarrow \|f\|_{Z^*} \leq 1$$

$$= \|x\|_Z$$

$$\|x\| = \sup_{\|f\|_{Z^*} \leq 1} |\langle f, x \rangle|$$

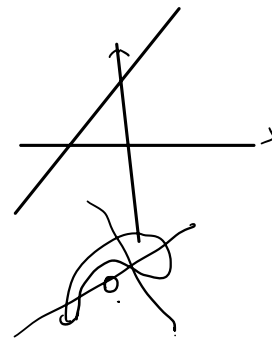
$X \rightarrow Y$ $B(X, Y)$ $B(X, Y)$ 完备 $\Rightarrow Y$ 完备

§ 12. 几何形式之 H-B 定理. 欧几里得几何定理

Def 2. Z 上之一子空间的定义为 $H = \{x \in Z : f(x) = d\}$.

eg. \mathbb{R}^2 . $f(a\vec{e}_1 + b\vec{e}_2) = a f(\vec{e}_1) + b f(\vec{e}_2) = d$.

$f(x) = d \Leftrightarrow (a, b)$. 上述 \uparrow



Prop 1. $H = [f=d]$ 闭 $\Leftrightarrow f$ 连续

Pf: " \Leftarrow " 显然

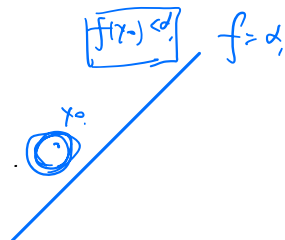
" \Rightarrow " 不妨假设 $d=0$ $[f=0]$ 闭 $\Rightarrow [f=d]$ 闭

$\forall x \in Z, f(x) \neq d, \forall f(x_0) = d, f(x-x_0) \neq 0$

$\exists B(x-x_0, \delta)$. s.t. $f(B(x-x_0, \delta)) \neq 0$. \forall

$$f(B(x, \delta)) \neq f(x_0) = \alpha$$

$\forall x_0 \in H^c$. $f(x_0) \neq \alpha$. 不妨假设 $f(x_0) < \alpha$
 $B(x_0, r) \subset H^c$.



$$f(x) < \alpha, \forall x \in B(x_0, r)$$

证. 若 $\exists x_1 \in B(x_0, r)$. $f(x_1) > \alpha$. $\exists \bar{x} \in \overline{x_0 x_1}$, s.t. $f(\bar{x}) = \alpha$. 矛盾.

$$f(x_0 + rz) < \alpha, \forall z \in B(0, 1)$$

$$\|f\| \leq \frac{1}{r}(\alpha - f(x_0))$$

证. 假设 f 不连续, $\exists x_n$. s.t. $|f(x_n)| \geq n \|x_n\|$

$$y_n = \frac{x_n}{f(x_n)}. \text{ 则 } f(y_n) = 1, \Rightarrow \|y_n\| \leq \frac{1}{n}, y_n \rightarrow 0.$$

又由 $[f=1]$ 闭. $\Rightarrow f(0) = 1$. 矛盾. #

Def 2. $A, B \subset \mathbb{R}$.

称 $[f=\alpha]$ 分离 A, B . 若 $f(x) \leq \alpha, \forall x \in A, f(y) \geq \alpha, \forall y \in B$.

严格分离 $\Rightarrow \exists \epsilon$. $f(x) \leq \alpha - \epsilon, \forall x \in A, f(y) \geq \alpha + \epsilon, \forall y \in B$.

Thm. 1 (超集分离定理) $A, B \subset \mathbb{R}$. 且 A 开, 则存在超集的分离 A, B .

Def 3. (Minkowski functional) $C \subset \mathbb{R}$, \mathbb{R}^n 开凸集.

$$p(x) = \inf \left\{ \alpha > 0 : \frac{x}{\alpha} \in C \right\}$$

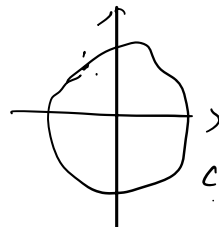
$$(1) C = \left\{ x \in \mathbb{R}^n : p(x) < 1 \right\}$$

$$(2) \exists M. \text{ s.t. } 0 \leq p(x) \leq M \|x\|$$

$$\exists B(0, r) \subset C, p(x) \leq \frac{1}{r} \|x\|, \forall x \in \mathbb{R}^n.$$

$$p(x) \leq \frac{1}{r} \|x\| \Leftrightarrow \frac{x}{\frac{1}{r} \|x\|} \in C \Leftrightarrow r \frac{x}{\|x\|} \in C.$$

(3) $p(x)$ 是次线性泛函.



Lemma. $C \subset \mathbb{R}^n$, 非空开凸集, $\mathbb{R}^n \setminus C \neq \emptyset$. $x_0 \notin C$. $\exists f \in E^*$. s.t. $f(x) < f(x_0), \forall x \in C$

Pf. $\mathbb{R}\{x_0\}$. 定义 $g(u(x_0)) = t \leq p(u(x_0))$. $\forall x_0 \in C$

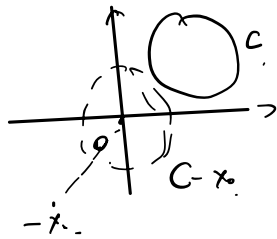
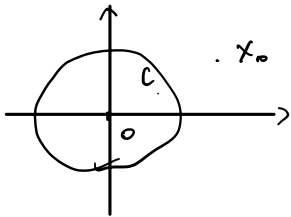
$$g(x_0) = 1 \leq p(x_0) \Rightarrow g(u(x_0)) \leq p(u(x_0)) \leq M \|u(x_0)\|$$

由 H-B Thm, $\exists f \in E^*$ s.t. $f(x_0) = g(x_0) = 1$.

$$f(x) \leq p(x) < 1, x \in C$$

f 即为所求. \square

特别地, $0 \notin C$, $\exists f \in E^*$ s.t. $f(x) < f(0) = 0, \forall x \in C$.



$$\forall x_0 \in C, C-x_0$$

$$0 \rightarrow -x_0$$

$$\exists f \in E^*$$

$$\text{s.t. } f(C-x_0) < f(-x_0)$$

$$f(C) < 0$$

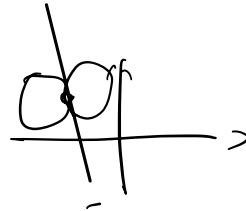
$$A-B = \{a-b \mid a \in A, b \in B\} = \bigcup_{b \in B} (A-b)$$

Proof of Thm 1. 令 $C = A-B$, 凸, 开, 由 $A \cap B = \emptyset \Rightarrow 0 \notin C$.

$$\exists f \in E^* \text{ s.t. } f(x) < 0, \forall x \in C$$

$$f(x-y) < 0, \forall x \in A, y \in B$$

$$f(x) < f(y), x \in A, y \in B$$



\exists 超平面 $[f=0]$ 分离 A, B .

Thm 2. (严格分离定理). $A, B \subset \mathbb{R}^n, A \cap B = \emptyset, A$ 闭, B 开. 则存在超平面严格分离 A, B .

Pf: $A-B$ 凸, 闭 ($\forall a-b_1 \rightarrow x, b_1 \in B \rightarrow b, a \in A \rightarrow x+b, x+b \in A, x \in A-B$)

$$0 \notin A-B. \text{ 存在 } B(b_0, r). \text{ s.t. } B(b_0, r) \cap (A-B) = \emptyset$$

则存在 $f \in E^*, [f=0]$ 分离 $B(b_0, r)$ 与 $A-B$.

$$f(a-b) \leq f(rz), \forall z \in B(b_0, r), a \in A, b \in B$$

$$f(a-b) \leq -r \|f\|, \text{ 令 } \varepsilon = \frac{1}{2} r \|f\|. f(x) + \varepsilon \leq f(y) - \varepsilon$$

Remark. E 内 ρ 凸, A, B 凸, $A \cap B \neq \emptyset$ 可分离

非 ρ 凸 ?

Cor 1. Z 是闭集, $F \subset Z$ 是子集. $\bar{F} \neq Z$, 则存在 $f \in Z^*$, $f \neq 0$, 且 $f|_F = 0$.

Remark, $\forall f \in Z^*$ $f|_F = 0 \Rightarrow f = 0$. 则 $\bar{F} = Z$.

Pf. $\forall x_0 \in \bar{F}$. $\{x_0\}$ 是 \bar{F} 闭集, $\exists f \in Z^*$ s.t. $f(x_0) - \varepsilon > f(x) + \varepsilon, \forall x \in F$.

$$f(x) < \overline{f(x_0)}, \forall x \in \bar{F} \text{ 是 } F \text{ 子集} \Rightarrow f|_{\bar{F}} = 0$$

($\exists y \in \bar{F}$ $f(y) \neq 0$ $f|_{\bar{F}} \rightarrow 0$ 矛盾)

eg. $\overline{L^\infty(\Omega)} \text{ 在 } W^{-k, p}(\Omega) = W^{-k, p}(\Omega)$

其中 $W^{-k, p}(\Omega) := (W^{k, p}(\Omega))^*$ ($1 < p < \infty$)

$$\|u\|_{W^{-k, p}(\Omega)} := \sup_{\|v\| \leq 1, v \in W^{k, p}(\Omega)} |\langle u, v \rangle|$$

1. $L^\infty(\Omega) \subset W^{-k, p}(\Omega) = (W^{k, p}(\Omega))^*$

fix $\varphi \in L^\infty(\Omega)$ $\forall u \in W^{k, p}(\Omega)$ $\langle \varphi, u \rangle = \int_\Omega \varphi u$

φ 有界? $\forall u_n \xrightarrow{W^{k, p}(\Omega)} 0$ $\langle \varphi, u_n \rangle \rightarrow 0$

$$|\langle \varphi, u_n \rangle| \leq \|u_n\|_p \|\varphi\|_{p'} \rightarrow 0, n \rightarrow \infty$$

2. $L^\infty(\Omega) \not\subset W^{-k, p}(\Omega)$ 子集.

$W^{k, p}(\Omega)$ 是 k . $(W^{k, p}(\Omega))^{**} = \overline{W^{k, p}(\Omega)}$

$\forall f \in (W^{-k, p}(\Omega))^* = (W^{k, p}(\Omega))^{**}$, $\exists u_f \in W^{k, p}(\Omega)$

$$\langle f, v \rangle = \langle v, u_f \rangle, \forall v \in W^{-k, p}(\Omega)$$

若 f 作用在 $L^\infty(\Omega)$ 上恒为 0. $\forall \varphi \in L^\infty(\Omega)$ $\langle f, \varphi \rangle = 0$

$$\langle \varphi, u_f \rangle = 0, u_f \in W^{k, p}(\Omega)$$

$$\Rightarrow u_f = 0, f = 0 \quad \#$$

$$W^{k,p}(\Omega) \xrightarrow{\cong} \left((L^p)^{\sim}(\Omega) \right) \quad L^p(\Omega) \rightarrow (L^p)^{\sim} \text{ h.k.}$$

$$u \mapsto (u, D^\alpha u, D^{\alpha_2} u, \dots, D^{\alpha_k} u) \quad \alpha_i \quad |F|=1$$

$$u \leftarrow u=1 \quad D_{x_i} u = 1$$

$$\tau: W^{k,p}(\Omega) \rightarrow (L^p)^{\sim}(\Omega)$$

$$(L^p)^{\sim} \text{ h.k.} \Rightarrow \tau(W^{k,p}(\Omega)) \text{ h.k.}$$

$$u \mapsto (u, \dots, D^{\alpha_k} u)$$

$$(L^p)^{\sim} \text{ in } \mathbb{R}^n$$

$$\tau: W^{k,p}(\Omega) \rightarrow \underbrace{\left(\tau(W^{k,p}(\Omega)) \right)}_{\text{h.k.}}$$

$$W^{k,p}(\Omega) \supset W_0^{k,p}(\Omega)$$

$$(W^{k,p}(\Omega))^* \subset (W_0^{k,p}(\Omega))^*$$

$$\underline{C_c^\infty(\Omega)} \subset \underline{(W^{k,p}(\Omega))^*}$$

$$\underline{(C_c^\infty(\Omega))} \left(W^{k,p}(\Omega) \right)^* = \boxed{(W_0^{k,p}(\Omega))^*}$$

Baire 定理及应用

Def. (Baire) 称 (X, τ) 是 Baire 空间, 若任意可列多个稠密开集的交仍为稠密. (任意可列多个无内点之闭集之并无内点).

Thm. (Baire) 完备度量空间 是 Baire 空间

证明: 设 $\{O_n\}$ 稠密开. 证 $\bigcap O_n$ 稠密.

$$\forall B(x, r), B(x, r) \cap (\bigcap O_n) \neq \emptyset$$

由 O_1 稠密, $B(x, r) \cap O_1 \neq \emptyset \Rightarrow \exists x_1 \in \bar{B}_1 \subset B(x, r) \cap O_1, \text{diam}(B_1) < \frac{r}{2}$

又 O_2 稠密, $B_1 \cap O_2 \neq \emptyset \Rightarrow \exists x_2 \in \bar{B}_2 \subset B_1 \cap O_2, \text{diam}(B_2) < \frac{r}{4}$

... 一直下去, $\{\bar{B}_n\}$ diam $(\bar{B}_n) \rightarrow 0, \bar{B}_n \downarrow$

$$\bigcap \bar{B}_n \neq \emptyset, \exists x \in \bigcap \bar{B}_n, x \in \bigcap O_n \cap B(x, r)$$

$$\Rightarrow B(x, r) \cap (\bigcap O_n) \neq \emptyset$$

Remark: 一个度量空间不完备, 它是否是 Baire 空间?

例:

$$\phi(x) = \frac{x}{1+|x|}, x \in \mathbb{R}$$

$$d(x, y) = |\phi(x) - \phi(y)|$$

d 不完备

$$(\mathbb{R}, \tau_d) = (\mathbb{R}, \tau)$$

Thm. 局部仿 Euclidean 空间也是 Baire 空间. X



证: $\{O_n\}$ 稠密开. $\bigcap O_n$ 稠密. \forall 开集 U

$\forall x \in X$

U_x 是 x 的邻域

由 O_1 稠, $U \cap O_1 \neq \emptyset \Rightarrow \exists x_1 \in V_{x_1} \subset U \cap O_1$

$\forall x, x \in V_x \subset U_x$

O_2 稠, $V_{x_1} \cap O_2 \neq \emptyset \Rightarrow \exists x_2 \in V_{x_2} \subset V_{x_1} \cap O_2$

得到 $\{\bar{V}_{x_n}\}, V_{x_n} \downarrow, \bar{V}_{x_n} \cap \bar{V}_{x_{n+1}} \neq \emptyset, \bar{V}_{x_n} \cap \bar{V}_{x_{n+1}} \neq \emptyset$

$\{\bar{V}_{x_n}\}$ 闭子族, $\bigcap \bar{V}_{x_n} \neq \emptyset \Rightarrow x \in \bigcap \bar{V}_{x_n}$

$$x \in (\bigcap_n O_n) \cap U.$$

$$(\bigcap_n O_n) \cap U \neq \emptyset. \quad \#$$

Banach-Steinhaus Thm.

设 Z 是 Banach 空间, F 是赋范空间, $\{u_i\}_{i \in \mathbb{Z}} \subset \mathcal{B}(Z, F)$, $\forall x \in Z$.

$$\sup_{i \in \mathbb{Z}} \|u_i(x)\| < +\infty \quad \#$$

$$\sup_{i \in \mathbb{Z}} \|u_i\| < +\infty$$

$$\text{证明: } Z = \bigcup_{n=1}^{\infty} \left\{ x \in Z \mid \sup_{i \in \mathbb{Z}} \|u_i(x)\| \leq n \right\} = \bigcup_{n=1}^{\infty} F_n$$

$$F_n \text{ 闭. } F_n = \bigcap_{i \in \mathbb{Z}} \left\{ x \in Z \mid \|u_i(x)\| \leq n \right\}$$

$$\boxed{\exists F_n \neq \emptyset} \quad \left(\forall n, \forall x \in F_n, F_n \neq \emptyset, Z = \bigcup_{n=1}^{\infty} F_n \Rightarrow Z \text{ 是 } \sigma\text{-紧} \right)$$

$$\exists B(x, r). \quad B(x, r) \subset F_n \quad \forall y \in B(x, r)$$

$$\sup_{i \in \mathbb{Z}} \|u_i(y)\| \leq n$$

$$\forall z \in B(0, r) \quad \|u_i(z+x)\| \leq n \Rightarrow \|u_i(z)\| \leq \|u_i(x)\| + n = 2n$$

$$\|u_i\| \leq \frac{2n}{r} \quad \forall i \in \mathbb{Z}$$

$$\Rightarrow \sup_{i \in \mathbb{Z}} \|u_i\| \leq \frac{2n}{r}$$

Thm. Z 是 Banach, F 赋范. $\{u_i\}_{i \in \mathbb{Z}} \subset \mathcal{B}(Z, F)$. $\sup_{i \in \mathbb{Z}} \|u_i\| = +\infty$

则 $\left\{ x \in Z \mid \sup_{i \in \mathbb{Z}} \|u_i(x)\| < +\infty \right\}$ 是 Z 的真子集. $(0 \in G_s \Rightarrow 0 = \bigcap_n O_n)$

$$\text{证明: } \left\{ x \in Z \mid \sup_{i \in \mathbb{Z}} \|u_i(x)\| < +\infty \right\} = \bigcap_{n=1}^{\infty} \left\{ x \in Z \mid \sup_{i \in \mathbb{Z}} \|u_i(x)\| < n \right\} = \bigcap_{n=1}^{\infty} O_n$$

$$O_n \text{ 闭. } \left\{ x \in Z \mid \sup_{i \in \mathbb{Z}} \|u_i(x)\| \leq n \right\} = (O_n)^c = F_n \text{ 闭} \Rightarrow O_n \text{ 开}$$

$$\forall n, O_n \text{ 不相交. } F_n = O_n^c \text{ 有内点} \Rightarrow \sup_{i \in \mathbb{Z}} \|u_i\| < +\infty \text{ 矛盾} \Rightarrow O_n \text{ 不相交. } \#$$

命题: Z Banach, F 线性空间, $(u_n) \subset B(Z, F)$. 若 (u_n) 逐点收敛到 u .

则 $u \in B(Z, F)$ 且 $\|u\| \leq \liminf \|u_n\|$

$$u_n(x) \rightarrow u(x)$$

证明: $u \in B(Z, F)$, u 线性映射

$$\sup_n \|u_n\| < +\infty$$

$$\liminf_n \|u_n\| < +\infty$$

$$\|u(x)\| = \lim_n \|u_n(x)\| \leq \left(\liminf_n \|u_n\| \right) \|x\|$$

$$\Rightarrow \|u\| \leq \liminf_n \|u_n\| < +\infty$$

命题: Z Banach, $x_n \in Z$, $x_n \rightarrow x$ 且 $\|x\| \leq \liminf \|x_n\|$

证明: $x_n \rightarrow x \quad \forall f \in Z^* \quad \langle f, x_n \rangle \rightarrow \langle f, x \rangle$

$$\langle x_n^{**}, f \rangle \rightarrow \langle x^{**}, f \rangle$$

$\{x_n^{**}\} \subset Z^{**}$. $\{x_n^{**}\}$ 在 $\forall f \in Z^*$ 上有界, 且 $\sup_n \|x_n^{**}\| < +\infty$

$$\|x^{**}\| \leq \liminf_n \|x_n^{**}\|$$

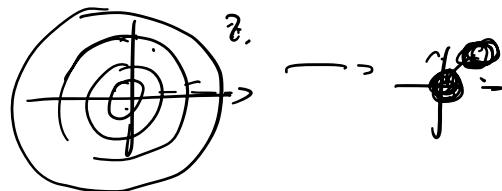
$$\sup_n \|x_n\| < +\infty$$

$$\Rightarrow \|x\| \leq \liminf_n \|x_n\|$$

Thm. (开映射定理) 设 Z, F Banach, $u \in B(Z, F)$ 满射, 则 u 开映射.

证明: ① $B_F \subset \overline{u(B_Z)}$, $\exists c > 0$

$$F = \bigcup_{n=1}^{\infty} u(B_n) = \bigcup_{n=1}^{\infty} \overline{u(B_n)} = \bigcup_{n=1}^{\infty} F_n$$



$\exists F_n, F_n \neq \emptyset$. ($\forall n, F_n = \emptyset \Rightarrow F$ 无点, 矛盾)

$$\exists B(y, \eta), \text{ s.t. } B(y, \eta) \subset \overline{u(B_{n_0})}$$

$$\eta B_F \subset \overline{u(B_{n_0})} - y_n \subset \overline{u(B_{2n_0})}$$

$$\Rightarrow \exists c, B_F \subset \overline{u(cB_Z)}$$

$$\textcircled{2} B_F \subset \overline{u(cB_Z)} \subset u(2cB_Z)$$

$$\forall y \in B_F, \exists x_0 \in B_Z, \|y - u(x_0)\| \leq \frac{1}{2}$$

$$y_1 = 2[y - u(x_0)], \|y_1\| \leq 1 \Rightarrow y_1 \in B_F, \exists x_1 \in B_Z$$

$$\text{s.t. } \|y_1 - u(x_1)\| \leq \frac{1}{2}$$

$$\dots, \{x_n\}, \{y_n\}$$

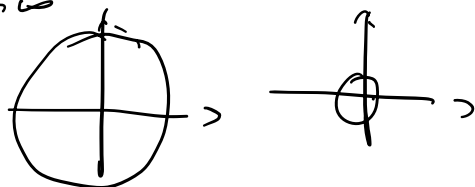
$$y = \frac{1}{2} y_1 + u(x_0) = u(x_0) + \frac{1}{2} y_1$$

$$= u(x_0) + \frac{1}{2} u(x_0) + \frac{1}{4} y_1$$

$$= u\left(\sum_{k=0}^n \frac{x_k}{2^k}\right) + \frac{1}{2^{n+1}} y_{n+1} \quad (*)$$

$$n \rightarrow \infty \quad y_n \in B_F, \|y_n\| \leq 1 \quad \rightarrow \quad \sum_{k=1}^n \frac{x_k}{2^k} \rightarrow x$$

$$\sum_{k=1}^n \frac{\|x_k\|}{2^k} \leq C \sum_{k=1}^n \frac{1}{2^k} \leq 2C, \quad n \rightarrow \infty$$



$$\exists x \in B_Z, \sum_{k=1}^n \frac{x_k}{2^k} \rightarrow x$$

$$\text{令 } n \rightarrow \infty, \quad y = u(x)$$

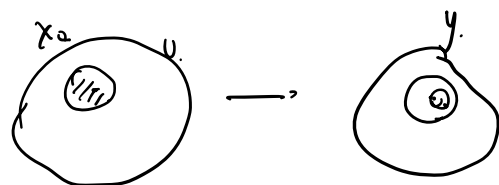
$$[B_F \subset u(B_Z)]$$

$$y \in B_F \subset u(B_Z)$$

$$③. \text{ 开映射 } f: U \rightarrow V, \quad u \in U, \quad f(u) = y$$

$$\forall y \in u(U), \quad x_0 \in U, \quad u(x_0) = y$$

$$\exists B(y, \eta) \subset u(U)$$



$$\forall U \subset Z, \quad f: U \rightarrow V$$

$$\forall y_0 \in u(U), \quad x_0 \in U, \quad u(x_0) = y_0$$

$$\forall \epsilon > 0, \quad B(x_0, \epsilon) \subset U, \quad x_0 + B(0, \epsilon) \subset U$$

$$y_0 + u(B(0, \epsilon)) \subset u(U)$$

$$u(B(0, \epsilon)) \supset B(0, \eta)$$

$$u(B(0, \epsilon)) \supset B(0, \eta)$$

$$B(y_0, \eta) \subset u(U)$$

映射 (逆映射定理). Z, F Banach, $u \in \mathcal{B}(Z, F)$. 双射 $\Rightarrow u^{-1} \in \mathcal{B}(F, Z)$

$$\tau: (Z, \|\cdot\|_1) \rightarrow (Z, \|\cdot\|_2)$$

$$x \mapsto x \quad \text{双射} \quad \|x\|_2 \leq c \|x\|_1$$

$$\exists c^{-1} \|x\|_1 \leq c \|x\|_2$$

Thm (闭图定理) Z, F Banach 空间. $u \in \mathcal{L}(Z, F)$, 则

u 连续 $\Leftrightarrow G(u)$ 闭

$$G(u) = \{ (x, u(x)) \mid x \in Z \}$$

证明: " \Rightarrow " $\forall x_n \rightarrow x, u(x_n) \rightarrow u(x), (x, u(x)) \in G(u)$

" \Leftarrow ": $G(u)$ 闭. $G(u)$ 是 $Z \times F$ 上之闭的赋范空间, $Z \times F$.

$$(x, y) \in Z \times F. \quad \|(x, y)\|_{Z \times F} = \|x\|_Z + \|y\|_F$$

$G(u)$ 是 Banach 空间.

$P: G(u) \rightarrow Z, (x, u(x)) \mapsto x$ 双射.

$$\|x\| \leq \|(x, u(x))\| = \|x\| + \|u(x)\|$$

P 连续. $\|P\| \leq 1$

P^{-1} 连续. $\|(x, u(x))\| \leq c \|x\| \Rightarrow \|u(x)\| \leq c \|x\|$

$\Rightarrow u$ 连续.

eg 1. Z Banach, $T: Z \rightarrow Z^*$. 对称性. T 自伴. $\langle Tx, y \rangle = \langle Ty, x \rangle$

$\forall x, y \in Z$, 则 T 连续.

证明: $G(T)$ 闭. $\Leftrightarrow \begin{cases} x_n \rightarrow x \text{ in } Z. \\ Tx_n \rightarrow f \text{ in } Z^*. \end{cases} \Rightarrow f = Tx$

$$\langle Tx_n, y \rangle = \langle Ty, x_n \rangle \quad \forall y \in Z.$$

$$x_n \rightarrow x \quad Tx_n \rightarrow f \text{ in } Z^*$$

$$\langle Tx_n, y \rangle \rightarrow \langle f, y \rangle$$

$$\langle Ty, x_n \rangle \rightarrow \langle Ty, x \rangle$$

$$\Rightarrow \langle f, y \rangle = \langle Ty, x \rangle = \langle Tx, y \rangle$$

$$\Rightarrow \langle f - Tx, y \rangle = 0 \quad \forall y \in Z \quad \Rightarrow f = Tx$$

eg2. Z, F Banach. $u: Z \rightarrow F$ 线性

(a) G 是 Hausdorff 空间 $v: F \rightarrow G$ 连续, 单射. u 连续 $\Leftrightarrow v \circ u$ 连续
 \Rightarrow " 显然

$$\Leftarrow: \text{Graph}(u) \begin{cases} x_n \rightarrow x \text{ in } Z \\ u(x_n) \rightarrow y \text{ in } F \end{cases} \Rightarrow u(x) = y$$

$$u(x_n) \rightarrow y \text{ in } F$$

$$v \text{ 连续 } \Rightarrow v \circ (u(x_n)) \rightarrow v(y)$$

$$\Rightarrow v(y) = v(u(x)) \Rightarrow y = u(x)$$

$$v \circ u(x_n) \rightarrow v(u(x))$$

(b) u 连续 \Leftrightarrow 当 F 上拓扑比原拓扑更强为 Hausdorff 拓扑, u 连续

$$(F, \tau_F), \quad G = (F, \tau_H) \quad \tau_H \subset \tau_F$$

$$v: (F, \tau_F) \rightarrow (F, \tau_H) \quad x \mapsto x, \quad \text{单射, 连续}$$

$$v \circ u \text{ 连续} \quad (Z, \tau_Z) \rightarrow (F, \tau_H) \text{ 连续}$$

$$\Rightarrow (Z, \tau_Z) \rightarrow (F, \tau_F) \text{ 连续}$$

$$\sigma(F, F^*)$$

$$u: (Z, \|\cdot\|_Z) \rightarrow (F, \sigma(F, F^*)) \text{ 连续, 则}$$

$$u: (Z, \|\cdot\|_Z) \rightarrow (F, \|\cdot\|_F) \text{ 连续}$$

is: 泛函, $G(u)$ 泛函.

$$\begin{cases} x_n \rightarrow x \text{ in } Z \\ u(x_n) \rightarrow y \text{ in } F \end{cases} \Rightarrow y = u(x)$$

$x_n \rightarrow x$ in Z . $\Rightarrow u(x_n) \rightarrow u(x)$ weakly

$$\forall f \in F^*, \langle f, u(x_n) \rangle \rightarrow \langle f, u(x) \rangle$$

$$\langle f, u(x_n) \rangle \rightarrow \langle f, y \rangle$$

$$\Rightarrow \langle f, u(x) \rangle = \langle f, y \rangle \quad \forall f \in F^*$$

$$\langle f, u(x) - y \rangle = 0 \quad \forall f \in F^* \quad x \in \bar{Z} \quad x \neq 0 \quad \exists f \in F^* \quad f(x) \neq 0$$

$$\Rightarrow u(x) - y = 0 \quad \Rightarrow y = u(x)$$

可赋范空间.

$$\Omega \subset \mathbb{R}^n \text{ bdd. } C(\bar{\Omega}) \quad \|f\| = \sup_{x \in \bar{\Omega}} |f(x)|$$

$$C(\Omega) \quad f_n \rightarrow f \text{ in } C(\Omega)$$

$$\Leftrightarrow \forall K \subset \subset \Omega, \sup_{x \in K} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty$$

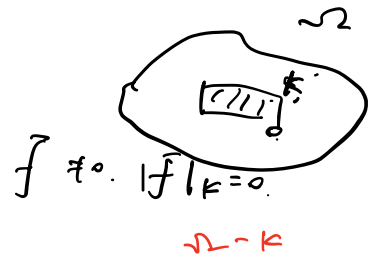
$$\|f\|_K := \sup_{x \in K} |f(x)|, \quad K \subset \subset \Omega$$

$$p(x) = 0 \Leftrightarrow x = 0 \quad x$$

$$\| \cdot \|_K \text{ 范数, } \mathbb{R} \text{ 范数}$$

$$\| \cdot \|_K$$

$$K \subset \subset \Omega \quad \{ \| \cdot \|_K \} \rightarrow \tau$$



Def. 范数范数空间. $p: Z \rightarrow [0, +\infty)$

$$(1) p(x) \geq 0. \quad (2) p(\lambda x) = |\lambda| p(x), \quad \forall x \in Z, \lambda \in \mathbb{K}$$

$$(3) p(x+y) \leq p(x) + p(y) \quad (\text{范数})$$

Remark: $d(x, y) := p(x - y)$ $B(x, \epsilon) = \{y \in \mathbb{R} \mid d(x, y) < \epsilon\} \Rightarrow$ 拓扑, τ_p

τ_p Hausdorff $\Leftrightarrow p$ 是 \mathbb{R} 上的范数

$p(x) = 0 \Rightarrow x = 0$ $x \neq 0, p(x) = 0$ $x = 0$

Def. $(p_i)_{i \in I}$ 是 \mathbb{R} 上的一族范数, 定义 $(p_i)_{i \in I}$ 诱导的拓扑 τ ,

$0 \in \tau \Leftrightarrow 0 = \bigcup_{\alpha \in I} B_{p_\alpha}(x_\alpha, r_\alpha)$

α 为指标集. $J_\alpha \subset I$, 有限

$B_{p_{J_\alpha}}(x_\alpha, r_\alpha) := \{y \in \mathbb{R} \mid \max_{i \in J_\alpha} p_i(x_\alpha - y) \leq r_\alpha\}$

τ 是一族拓扑, $\phi: x \in \tau$, 且任意并封闭, 且有限交也有有限交封闭.

$\bigcap_{i=1}^n \bigcup_{\alpha \in \Lambda_i} B_{p_{J_\alpha}}(x_\alpha, r_\alpha)$ $\Lambda_1, \dots, \Lambda_n$ 指标集.

$\forall \alpha_i \in \Lambda_i$ $\bigcap_{i=1}^n B_{p_{J_{\alpha_i}}}(x_{\alpha_i}, r_{\alpha_i})$

$\forall x \in \bigcap_{i=1}^n B_{p_{J_{\alpha_i}}}(x_{\alpha_i}, r_{\alpha_i})$

$x \in B_{p_{J_{\alpha_i}}}(x_{\alpha_i}, r_{\alpha_i}) \Leftrightarrow \forall p \in p_{J_{\alpha_i}} \quad p(x - x_{\alpha_i}) < r_{\alpha_i}$

若 $\tilde{J} = J_{\alpha_1} \cup \dots \cup J_{\alpha_n}(x, \epsilon)$. $\epsilon > 0 \exists \eta: B_{p_{\tilde{J}}}(x, \epsilon) \subset \bigcap_{i=1}^n B_{p_{J_{\alpha_i}}}(x_{\alpha_i}, r_{\alpha_i})$

$\forall y \in B_{p_{\tilde{J}}}(x, \epsilon)$ $\forall p \in p_{J_{\alpha_i}} \quad p(x - x_{\alpha_i}) < r_{\alpha_i}$

$p(y - x_{\alpha_i}) \leq p(y - x) + p(x - x_{\alpha_i}) \leq \epsilon + p(x - x_{\alpha_i}) < r_{\alpha_i}$

$\bigcap_{i=1}^n B_{p_{J_{\alpha_i}}}(x_{\alpha_i}, r_{\alpha_i})$ 非空.

Remark. $\{p_i\}_{i \in I}$ $p_J \subset I$. $p_{p_J}(x, \epsilon)$

$\exists \{p_i\}_{i \in I} \forall x \in \mathbb{R}$. $\forall p_1, p_2, \exists p_3$ s.t. $p_3 \geq \max\{p_1, p_2\}$

$\forall J \subset I$ $p \in I$ $B_{p_1 \cup p_2}(x, \epsilon) \supset B_{p_3}(x, \epsilon)$ $B_{p_i}(x, \epsilon)$

$\forall x \neq 0, \exists p_i(x) \neq 0$

Remark. (p_i) is a family of Hausdorff $\Rightarrow p_i$ is \leq $\forall x \neq y, \exists p_i(x) \neq p_i(y)$

$\Leftrightarrow \forall x \neq y, x \neq 0, \exists B_{p_j}(x, \varepsilon) \neq \emptyset$

$p_i = |f_i|$

$\forall p \in \mathcal{P}, p(x) \geq \varepsilon, p(x) \neq 0$

$\Leftrightarrow \forall x \neq 0, \exists p_i(x) \neq 0$

$\forall x \neq y, x, y \neq 0, x - y \neq 0, \exists p_i$ s.t. $p_i(x - y) \neq 0$

\Rightarrow condition $B_{p_i}(x, \varepsilon) \cap B_{p_i}(y, \varepsilon) = \emptyset$

$p_i(z - x) \leq \varepsilon, p_i(z - y) \leq \varepsilon \Rightarrow p_i(x - y) \leq 2\varepsilon$ (by triangle inequality)

Def (topological vector space). V is a vector space on K . τ is a topology on V .

$$\tau = \left\{ \begin{array}{l} z \times z \rightarrow z \\ \underline{(x, y)} \mapsto x + y \end{array} \right. \quad \tau = \left\{ \begin{array}{l} K \times z \rightarrow z \\ \underline{(a, x)} \mapsto \lambda x \end{array} \right.$$

connected. then V is a topological vector space.

Thm. Let (p_i) be a family of seminorms on V , then $\tau(p_i)$ is a topology on V (if V is connected) and this topology is the coarsest topology such that each p_i is continuous.

Pf. (p_i) is continuous in τ .

$\forall x \in B_{p_i}(x_0, \varepsilon), |p_i(x) - p_i(x_0)| \leq p_i(x - x_0) < \varepsilon \Rightarrow p_i$ is continuous.

τ is the coarsest topology such that each p_i is continuous. Let $\tau' \subset \tau$.

$\mathcal{N}_{\tau}(0) \subset \mathcal{N}_{\tau'}(0)$. For any $V \in \mathcal{N}_{\tau}(0)$ there exists $U \in \mathcal{N}_{\tau'}(0)$ s.t. $U \subset V$.

For any $U \in \mathcal{N}_{\tau'}(0)$, s.t. $B_{p_j}(0, \varepsilon) \subset U$.

$B_{p_j}(0, \varepsilon) = \bigcap_{i \in \mathcal{I}} B_{p_i}(0, \varepsilon)$, p_i is continuous in τ' . $B_{p_i}(0, \varepsilon) \in \mathcal{N}_{\tau'}(0)$

$B_{p_j}(0, \varepsilon) \in \mathcal{N}_{\tau}(0)$ \square

Thm. 可分离范数空间是局部凸的, 即存在一族凸邻域基

pf. 对邻域基 $\{B_{\rho_j}(0, \epsilon_j)\}$

$$(E, \|\cdot\|) \rightarrow (F, \|\cdot\|)$$

$$\exists c > 0, \|u(x)\| \leq c \|x\|$$

Thm. 设 $(E, \rho_i)_{i \in I}$, $(F, \rho_j)_{j \in J}$, $u: E \rightarrow F$ 线性.

u 连续 \Leftrightarrow 对 $\forall j \in J$ 存在有限集 I' 及 $C > 0$ s.t.

$$\max_{j \in J'} \rho_j(u(x)) \leq C \max_{i \in I'} \rho_i(x) \quad \forall x \in E.$$

pf: \Rightarrow " u 连续, $u^{-1}(B_{\max_{j \in J'} \rho_j(u, 1)})$ 是 E 中开集, $\exists I'$.

$$B_{\max_{i \in I'} \rho_i(x, r)} \subset u^{-1}(B_{\max_{j \in J'} \rho_j(u, 1)})$$

$$\text{即当 } \max_{i \in I'} \rho_i(x) < r \text{ 时, } \max_{j \in J'} \rho_j(u(x)) < 1.$$

$$\max_{i \in I'} \rho_i(x) < r$$

$$\forall x \in E, \delta > 0, \max_{j \in J'} \rho_j(u(x)) < 1 \text{ 当 } \max_{i \in I'} \rho_i(x) < \frac{\delta}{C}$$

$$\max_{j \in J'} \rho_j(u(x)) < \frac{\max_{i \in I'} \rho_i(x) + \delta}{C}, \quad \delta \rightarrow 0, \quad C = \frac{1}{r}$$

\Leftarrow " $\forall V \in \mathcal{N}(0)$, 存在 $J' \subset J$ 有限集及 $\epsilon > 0$ s.t. $B_{\max_{j \in J'} \rho_j(u, \epsilon)} \subset V$.

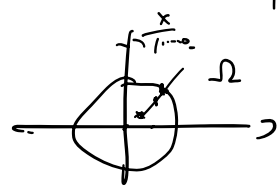
$$\exists I' \subset I \text{ 及 } c > 0, \quad r = \frac{\epsilon}{c} \text{ 时, } B_{\max_{i \in I'} \rho_i(x, r)} \subset u^{-1}(B_{\max_{j \in J'} \rho_j(u, \epsilon)})$$

$$u^{-1}(V) \in \mathcal{N}(0) \Rightarrow u \text{ 连续}$$

二、局部凸拓扑的范数空间 \Rightarrow 可分离范数

Minkowski 范数 ρ

$$0 \in \Omega, \text{ 开集, } \rho_\Omega(x) = \inf \left\{ \lambda > 0 \mid \frac{x}{\lambda} \in \Omega \right\}$$



$$x \in \Omega \Leftrightarrow \rho_\Omega(x) < 1$$

$$x \in \partial \Omega, \rho_\Omega(x) = 1$$

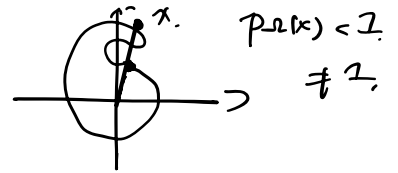
$$\Omega \text{ 平衡集 } \forall x \in \Omega, \lambda \in [0, 1] \Rightarrow \lambda x \in \Omega$$

$$\Rightarrow \Omega = \{x : \rho_\Omega(x) < 1\}$$

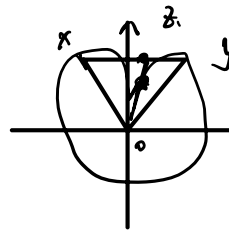
$$p_{\Omega}^{-1}((t, t, 1)) = \{ p_{\Omega}^{-1}((t, t, 1)) \} = \underline{\Omega}$$

$$\forall x \in \Omega, |a| \leq 1, \lambda x \in \Omega$$

Ω 开集 $\Rightarrow \varepsilon \Omega$ 开集 $\Rightarrow p_{\Omega}$ 连续



Ω 闭集 $\Rightarrow p_{\Omega}$ 连续 \equiv 闭映射



$$p_{\Omega}(x) = 1$$

$$p_{\Omega}(y) = 1$$

$$p_{\Omega}(z) > 1$$

$$z = tx + u + iy$$

$$p_{\Omega}(z) \leq t p_{\Omega}(x) + (1-t) p_{\Omega}(y)$$

$$\leq 1 \text{ 矛盾}$$

闭集: 平行于 Ω 的闭包域基

Lemma. 拓扑向量空间的闭包域基有平行于 Ω 的闭包域基

pf. 对于 $V \in \mathcal{N}(W)$ 寻找 $U \subset V$ 平行于 $U \in \mathcal{N}(W)$

$$\Psi: \mathbb{K} \times V \rightarrow V \quad (\lambda, x) \mapsto \lambda x, \quad \Psi(0, 0) = 0$$

$$\exists B_{\mathbb{K}}(0, \delta) \text{ 及 } V' \in \mathcal{N}(W). \text{ s.t. } \Psi(B_{\mathbb{K}}(0, \delta) \times V') \subset V$$

$$\text{即 } \forall 0 < \lambda < \delta, \lambda V' \subset V$$

$$\text{令 } U = \bigcup_{|\lambda| < \delta} \lambda V' \subset V. \text{ 平行于 } \forall \mu \in \mathbb{K}, (|\mu| \leq 1)$$

$$\mu U \subset U, \quad \mu U = \bigcup_{|\lambda| < \delta} \mu \lambda V' \quad (|\mu| \leq 1, |\mu \lambda| < \delta)$$

$$\subset \bigcup_{|\lambda| < \delta} \lambda V' = U$$

$\Rightarrow U$ 平行于 $U \in \mathcal{N}(W)$ 开

Thm. 拓扑向量空间的闭包域基有平行于 Ω 的闭包域基

$$\text{pf. } \{U_{\alpha}\}_{\alpha \in \Lambda}, \quad \{\text{conv}(U_{\alpha})\}_{\alpha \in \Lambda}$$

$$\forall V \in \mathcal{N}(W), \exists A \in \mathcal{N}(W), A \subset V, A \text{ 凸}$$

$$\exists B \in \mathcal{N}(W), \text{ s.t. } B \subset A, B \text{ 平行于开包域}$$

$$\text{conv}(B) = \left\{ \sum_{i=1}^n \lambda_i b_i; 0 \leq \lambda_i \leq 1, \sum_{i=1}^n \lambda_i = 1, b_i \in B \right\}$$

$$A \text{ 凸 } B \subset A, \text{ conv}(B) \subset A, \text{ conv}(B) \text{ 平行于}$$

$$(\text{conv}(B))^{\circ} : \text{开集, 凸集, 平衡集} \quad (\text{conv}(K))^{\circ} \subset V$$

def. (Minkowski). Ω 开, 凸, 平衡集. $\Omega \in \mathcal{N}(W)$, $p_{\Omega} : \mathbb{R} \rightarrow \mathbb{R}$

$$p_{\Omega}(x) = \inf \{ \lambda > 0 : \frac{x}{\lambda} \in \Omega \}$$

prop. (1) $x \in \Omega \Leftrightarrow p_{\Omega}(x) \leq 1, \Leftrightarrow \Omega$ 平衡集

(2) $\Omega_1 \subset \Omega_2 \Rightarrow p_{\Omega_2} \leq p_{\Omega_1}$

(3) $\Omega_3 = \Omega_1 \cap \Omega_2, p_{\Omega_3} = \max\{p_{\Omega_1}, p_{\Omega_2}\}$.

验证: $\{p_{\Omega}\}$ 收敛.

Thm. τ 凸开集. $(p_{\Omega})_{\Omega}$ 收敛, Ω 是取遍 Ω 之凸平衡开子集.

且 p_{Ω} 收敛于 τ 与 τ 一致.

pf: $\Rightarrow \tau' \subset \tau$.

p_{Ω} 收敛, 在 τ' , $p_{\Omega}^{-1}(1-\epsilon, 1) = \{p_{\Omega}^{-1}(1, 1) = \epsilon\Omega\}$, 开集,

$\Rightarrow \tau \subset \tau'$

$\forall 0 \in \mathcal{N}(W), \exists U \in \mathcal{N}(W), 0 \in U$.

$0 \in \mathcal{N}(W), \exists \Omega \subset U, \Omega$ 凸, 平衡, 开.

$$\Omega = \{x \in U : p_{\Omega}(x) < 1\} \in \mathcal{N}(U)$$

$U \supset \Omega \subset U, \Omega \in \mathcal{N}(U), \Rightarrow \tau \subset \tau', \#$.

eg. 1. $C^{\infty}(a, b)$ $p_i(f) = \sup_{x \in [a, b]} |f^{(i)}(x)|$.

$\forall f \neq 0, p_i(f) = \sup_{x \in [a, b]} |f^{(i)}(x)| \neq 0, \Rightarrow \{p_i\}$ 可分.

$f_n \rightarrow f \Leftrightarrow \forall \epsilon, f_n^{(i)} \rightarrow f^{(i)}$ in $[a, b]$.

$\{p_i\}$ 收敛于 τ 且 τ 是凸集. $d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{p_k(x-y)+1}$

$$C^\infty([0,1]^d). \quad \rho_\epsilon(f) = \sup_{x \in [0,1]^d} |D^\alpha f(x)|, \quad \{\rho_\epsilon\} \text{ is a family, } \rho_\epsilon \text{ is a norm} \Rightarrow \text{is a norm}$$

eg 2. X is a Hausdorff space, $C(X, K)$

$$\text{is } K \subset X, \quad \rho_K(f) = \sup_{x \in K} |f(x)|$$

$\{\rho_K\}$ is a family, $f_n \rightarrow f \iff f_n \rightarrow f$ in K , $K \subset X$
 $\exists K_i \uparrow, \bigcup_{i=1}^\infty K_i = X, \{\rho_{K_i}\}$ is a family \Rightarrow is a norm

eg 3. $S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty\}$
 $(\|\cdot\|_{\alpha, \beta})$ is a family, \Rightarrow is a norm, not a norm

eg 4. $\Omega \subset \mathbb{R}^n$, bdd, $0 < p < \infty, L^p(\Omega)$
 $\forall K \subset \Omega, f \in L^p(K)$
 $\|f\|_p = \left(\int_K |f|^p \right)^{1/p}$
 $1 \leq p \leq \infty, \|\cdot\|$
 $0 < p < 1, \|\cdot\|$

$p \geq 1$ is a norm, \Rightarrow is a norm

$0 < p < 1$ is not a norm, not a norm

$$(\mathbb{R}, \|\cdot\|) \rightarrow (\mathbb{R}, \rho_K)$$

$$u: x \mapsto x \text{ is a map } \iff \forall \rho_K, \exists c, |\rho_K(x)| \leq c \|x\|$$

$$(\mathbb{R}, \rho_K) \rightarrow (\mathbb{R}, \|\cdot\|)$$

$$u: x \mapsto x \text{ is a map, } \exists \rho_1, \dots, \rho_n, 1 \leq i \leq n, \|x\| \leq c \max_{i=1, \dots, n} |\rho_i(x)|$$

$$\text{is a norm, } \|x\| \leq c |\rho_K(x)| \quad (\mathbb{R}, \rho_K) \text{ is Hausdorff}$$

$\Rightarrow (\mathbb{R}, \|\cdot\|)$ is a norm, ρ_K is a Hausdorff norm

Def. Z 是 Hausdorff 空间, Z^* 可分且 Z^* 诱导拓扑为弱拓扑.

prop. Z 是凸的, $(x_n) \subset Z$.

(i) $x_n \rightarrow x$ in $\sigma(Z, Z^*) \Leftrightarrow \forall f \in Z^*. \langle f, x_n \rangle \rightarrow \langle f, x \rangle$

(ii) $x_n \rightarrow x \Rightarrow x_n \rightarrow x$ in $\sigma(Z, Z^*)$

$|\langle f, x_n \rangle - \langle f, x \rangle| \leq \|f\| \|x_n - x\| \rightarrow 0$

(X, τ). τ 弱拓扑. $x_n \in X, x_n \rightarrow x, \exists N. \forall n > N, x_n = x$.

$\tau = \{ \phi, x \}. x_n$ 收敛

(iii) $x_n \rightarrow x$ in $\sigma(Z, Z^*)$, $(\|x_n\|)$ 有界 $\Rightarrow \|x\| \leq \liminf \|x_n\|$

$\forall f \in Z^*. f(x_n) \rightarrow f(x). \tau: Z \rightarrow Z^* x \mapsto \hat{x}. \langle \hat{x}, f \rangle = \langle f, x \rangle$

$\hat{x}_n \in Z^*$ 收敛. $\forall f \in Z^*. \langle \hat{x}_n, f \rangle = \langle f, x_n \rangle \rightarrow \langle f, x \rangle$

$\sup_n \| \hat{x}_n \|_{Z^*} < +\infty \Rightarrow \sup_n \| x_n \| < +\infty$

(iv) $x_n \rightarrow x$ in $\sigma(Z, Z^*)$, $f_n \rightarrow f$ in Z^* . $\Rightarrow \langle f_n, x_n \rangle \rightarrow \langle f, x \rangle$

$|\langle f_n, x_n \rangle - \langle f, x \rangle| \leq \|f_n - f\| \|x_n\| + |\langle f, x_n - x \rangle|$

Prop. 若 Z 有范数, 则 $(Z, \sigma(Z, Z^*)) = (Z, \|\cdot\|)$

Pf. $\tau_{\|\cdot\|} \subset \sigma(Z, Z^*)$

$\forall B(0, r)$. 存在 Z 的邻域 $V \subset B(0, r)$

$f(x_1 e_1 + \dots + x_n e_n) = x_1 \dots f_n(x_1 e_1 + \dots + x_n e_n) = x_n. f_1 \dots f_n \in Z^*$

$\{y: \max_{1 \leq i \leq n} |f_i(y)| < \epsilon\} \subset B(0, r)$

$|y_i| < \epsilon \Rightarrow |y| \leq C(\epsilon) \rightarrow 0$

Remark. 若 Z 有范数, 则 Z 的弱拓扑与范数拓扑一致.

Example 1. Z is a set. $\dim Z = +\infty$. $S = \{x \in Z : \|x\| = 1\}$. But S is not closed in $\sigma(Z, Z^*)$.

$$\overline{S}^{\sigma(Z, Z^*)} = B_Z. \quad (\text{闭单位球})$$

$$L^2([0,1]) \quad \sin 2n\pi x \rightarrow 0$$

Pf. $\forall x_0 \in Z, \|x_0\| < 1, x_0 \in \overline{S}^{\sigma(Z, Z^*)}$.

即证: 任意 x_0 在 $\sigma(Z, Z^*)$ 下 \in 闭单位球 V , $V \cap S \neq \emptyset$.

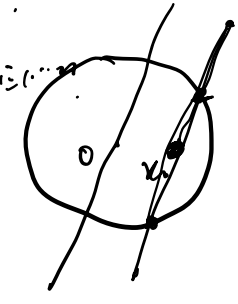
$$V = \{x \in Z : |f_i(x - x_0)| < \epsilon, i = 1, \dots, k, f_i \in Z^*\}$$

$\forall t \in \mathbb{R}$
 $x_0 + ty \in V$

Claim # $\exists y_0 \in Z, y_0 \neq 0, \langle f_i, y_0 \rangle = 0, \forall i = 1, \dots, k$.

若不然 $\varphi: Z \rightarrow \mathbb{R}^k$.

$$\varphi(x) = (f_1(x), f_2(x), \dots, f_k(x))$$



$\varphi(x) = 0 \Rightarrow x = 0$. $\varphi \neq 0 \Rightarrow \dim Z \geq \dim \mathbb{R}^k = k$. \sum 矛盾

$$\{x_0 + ty_0 : t \in \mathbb{R}\} \subset V. \quad x_0 + ty_0 \in V \Rightarrow |f_i(x_0 + ty_0 - x_0)| = |t f_i(y_0)| < \epsilon$$

$$\exists t \neq 0 \text{ s.t. } \|x_0 + ty_0 - x_0\| = \|ty_0\| < \epsilon \Rightarrow \|y_0\| < \epsilon / |t|$$

$\Rightarrow V \cap S \neq \emptyset$

$$\overline{S}^{\sigma(Z, Z^*)} \supset B_Z \supset S$$

$\forall x \notin B_Z, \{x\} \cap B_Z = \emptyset, \exists f \in Z^*$.

$$\underline{f(x) < \alpha \leq f(B_Z)}$$

\exists 闭球 B_Z 与 $\{x\}$ 不相交. $\Rightarrow B_Z$ 是闭的.

Example. $U = \{x \in Z : \|x\| < 1\}$ 在 $\sigma(Z, Z^*)$ 下不是闭的. $(U)^{\sigma(Z, Z^*)} = \emptyset$.

$$U^c = \{x \in Z : \|x\| \geq 1\}. \quad (\overline{U^c})^{\sigma(Z, Z^*)} = Z.$$

$$\Rightarrow (\overline{U})^{\sigma(Z, Z^*)} = \emptyset$$

problem: \mathbb{R} 上的 \neq 收敛性

\mathbb{R} 上的收敛性 $x_n \rightarrow x \Leftrightarrow \mathbb{R}$ 上的收敛性 $x_n \rightarrow x$

$$(x_n) \text{ 收敛} \Leftrightarrow \sum_n |x_n| < +\infty$$

收敛性的等价性 \Rightarrow 收敛性相同, X 不是 \mathbb{R}

(x, τ) , τ 收敛性, $\theta \in \tau \Leftrightarrow \theta \in \mathbb{Z}^*$ $x_n \rightarrow x \Leftrightarrow \exists N \forall n > N \forall x_n \rightarrow x$

τ 收敛性, $\theta \in \tau \Leftrightarrow \theta^c \in \mathbb{Z}^*$

$x_n \rightarrow x \Leftrightarrow \exists N \forall n > N \forall x_n \rightarrow x$

Thm (Mazur). $C \subset \mathbb{R}$. convex. C 在 $\sigma(\mathbb{R}, \mathbb{R}^*)$ 闭 $\Leftrightarrow \mathbb{Z}^*$ 收敛性

\Rightarrow 收敛性 $\rightarrow \mathbb{Z}^*$ 收敛性

\Leftarrow \mathbb{Z}^* 收敛性, $\overline{C}^{\sigma(\mathbb{R}, \mathbb{R}^*)} \supseteq C$. $\exists x \in \overline{C}^{\sigma(\mathbb{R}, \mathbb{R}^*)} \setminus C$

$\{x\}$ 凸, C 闭凸集, $\exists f \in \mathbb{R}^*$. $f(x) < \alpha < f(y)$, $\forall y \in C$

Cor. $(x_n) \subset \mathbb{R}$, $x_n \rightarrow x$. $\forall y \in \mathbb{R}$ $(x_n) \subset \mathbb{R}$ 收敛性 $\Rightarrow y \rightarrow x$

Pf. $C = \text{conv} \left(\bigcup_{n=1}^{\infty} \{x_n\} \right) \subset \mathbb{R}$. $x_n \rightarrow x \Rightarrow x \in \overline{C}^{\sigma(\mathbb{R}, \mathbb{R}^*)} = \overline{C}$

$\exists y_n \in C$. $y_n \rightarrow x$

$$\text{conv} A = \left\{ \sum_{i=1}^n \lambda_i a_i \mid \lambda_i \in \mathbb{R}, 0 < \lambda_i < 1, \sum_{i=1}^n \lambda_i = 1 \right\}$$

Thm. B.F. Remark. $T: \mathbb{R} \rightarrow \mathbb{F}$ (线性)

连续

$T: (\mathbb{R}, \|\cdot\|) \rightarrow (\mathbb{F}, \|\cdot\|)$ 连续 $\Leftrightarrow T: (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^*)) \rightarrow (\mathbb{F}, \sigma(\mathbb{F}, \mathbb{F}^*))$

Pf: \Rightarrow $x_n \rightarrow x$ in $\sigma(\mathbb{R}, \mathbb{R}^*)$

$T: \mathbb{R} \rightarrow \mathbb{F}$

$f \circ T: \mathbb{R} \rightarrow \mathbb{R}$

$T x_n \rightarrow T x$ in $\sigma(\mathbb{F}, \mathbb{F}^*)$

$f: \mathbb{F} \rightarrow \mathbb{R}$

$f \circ T \in \mathbb{R}^*$

$$\forall f \in \mathbb{F}^*. f(T x_n) = f \circ T(x_n) \rightarrow f \circ T(x) = f(T x)$$

$G(T)$ 闭, 在 $(Z \times F, \sigma(Z, Z^*) \times \sigma(F, F^*))$

$$x_u \rightarrow x \text{ in } \sigma(Z, Z^*), \quad Tx \rightarrow Tx \text{ in } \sigma(F, F^*)$$

$$\sigma(Z, Z^*) \times \sigma(F, F^*) = \sigma(Z \times F, (Z \times F)^*)$$

$G(T)$ 在 $(Z \times F, \sigma(Z \times F, (Z \times F)^*))$ 闭.

$G(T)$ 闭. $(x, Tx) \in G(T), (y, Ty) \in G(T) \Rightarrow (\lambda x + \mu y, \lambda Tx + \mu Ty) \in G(T)$

$\Rightarrow G(T)$ 在 $(Z \times F, \|\cdot\|_{Z \times F})$ 闭 $\|\cdot\|_{Z \times F} = \max\{\|\cdot\|_Z, \|\cdot\|_F\}$

$\Rightarrow T$ 连续. $(z, \| \cdot \|) \rightarrow (F, \| \cdot \|)$

双线性映射

def. Z 是赋范空间, $\tau: Z \rightarrow Z^{**}$ $x \mapsto \hat{x}$, $\hat{x}(f) = \langle f, x \rangle, \forall f \in Z^*$

则 $\{\|\hat{x}\|\}_{x \in Z}$ 是 Z^* 上的一致范数. 诱导范数 $\|\cdot\|_{Z^*}$ 是 Z^* 上的一致范数.

映射. 记为 $\sigma(Z^*, Z)$

$$\forall x_1, \dots, x_n \in Z. \quad B_{x_1, \dots, x_n}(f, \varepsilon) = \{g \in Z^* \mid |f(x_i) - g(x_i)| < \varepsilon, i=1, \dots, n\}$$

Remark. $Z^*, \|\cdot\|_{Z^*} \supset \sigma(Z^*, Z^{**}) \supset \sigma(Z^*, Z)$

Remark-Alaoglu Th. $Z^*, S = \{x \in Z^* \mid \|x\| = 1\}$ 是 $\|\cdot\|_{Z^*}$ 的, 是 Z^* 的闭球.

Prop. Z^* 是 Hausdorff.

$\{p_i\}_{i \in \mathbb{Z}}$ $\Rightarrow \tau$. τ Hausdorff $\Leftrightarrow \forall x \neq 0, \exists p: p(x) \neq 0$

$x \in Z, \{\|\hat{x}\|\}_{x \in Z}$. $\forall f \in Z^*, f \neq 0, \exists x \in Z, f(x) \neq 0$. 又 $f \neq 0$

$\Rightarrow \{\|\hat{x}\|\}_{x \in Z}$ 可分.

Prop. $(f_n) \subset Z^*$,

(i) $f_n \xrightarrow{*} f$ in $\sigma(Z^*, Z) \Leftrightarrow \langle f_n, x \rangle \rightarrow \langle f, x \rangle, \forall x \in Z.$

ii) If $f_n \rightarrow f \Rightarrow f_n \rightarrow f$ in $\sigma(Z^*, Z^{**})$

If $f_n \rightarrow f$ in $\sigma(Z^*, Z^{**}) \Rightarrow f_n \xrightarrow{*} f$ in $\sigma(Z^*, Z)$

$\forall \varphi \in Z^{**} \quad \varphi(f_n) \rightarrow \varphi(f) \quad \forall x \in Z. \quad \exists \tilde{x} \in Z^{**} \quad \tilde{x}(f_n) \rightarrow \tilde{x}(f)$

iii). $f_n \xrightarrow{*} f$ in $\sigma(Z^*, Z)$, $\|f_n\|$ bounded, and $\|f\| \leq \liminf \|f_n\|$

$\forall x \in Z. \quad \langle f_n, x \rangle \rightarrow \langle f, x \rangle$. $\{f_n\}$ weakly bounded $\Rightarrow \sup_n \|f_n\| < +\infty$

iv). $f_n \xrightarrow{*} f$ in $\sigma(Z^*, Z)$, $x_n \rightarrow x$ in Z . $\langle f_n, x_n \rangle \rightarrow \langle f, x \rangle$

$|\langle f_n, x_n \rangle - \langle f, x \rangle| \leq |\langle f_n, x_n \rangle - \langle f_n, x \rangle| + |\langle f_n, x \rangle - \langle f, x \rangle|$
 $\rightarrow 0 \qquad \qquad \qquad \rightarrow 0$

$x_n \rightarrow x \not\Rightarrow \langle f_n, x_n \rangle \rightarrow \langle f, x \rangle$

Remark. Z dual of Z $\dim(Z) < +\infty$ $Z: Z \rightarrow Z^{**}, x \mapsto \tilde{x}$

$\dim(Z^{**}) = \dim(Z)$. $Z \subset Z^{**}$. $\exists \tilde{x} \in Z^{**} \setminus Z$

$Z^{**} \rightarrow \mathbb{R}$

H.

prop.

$\varphi: Z^* \rightarrow \mathbb{R}$ linear, bounded, then $\exists x_0 \in Z$ s.t.

$\varphi(f) = \langle f, x_0 \rangle, \forall f \in Z^*$. $\textcircled{Z} \quad x_0 \in Z \subset Z^{**}$

pf: $Z \subset H$. $\forall x \in Z, \tilde{x}$

$H \subset Z^{**}$. $\varphi: (Z^*, H) \rightarrow \mathbb{R}$ linear $\Rightarrow \varphi \in Z^{**} \Rightarrow H \subset Z^{**}$

$Z \subset H \subset Z^{**}$

lem. X is normed space, $\varphi, \varphi_1, \dots, \varphi_k$ are X^* linear functionals. Γ is

$\sum_{i=1}^k \varphi_i(v) = 0, \forall v = (v_1, \dots, v_k) \in \mathbb{R}^k \Rightarrow \varphi(v) = 0$

$\Rightarrow \exists \lambda_1, \dots, \lambda_k \in \mathbb{R}$ s.t. $\varphi = \sum_{i=1}^k \lambda_i \varphi_i$

pf: $\exists \tilde{\varphi}_i \in X^{**}, \tilde{f}: X \rightarrow \mathbb{R}^{1 \times k}, u \mapsto [\varphi_1(u), \varphi_2(u), \dots, \varphi_k(u)]$

$(1, 0, \dots, 0) \notin \text{R}(\tilde{f})$

הו $\sum_{i=0}^k \lambda_i \varphi_i$ אזו $(\lambda_0, \dots, \lambda_k) \in \mathbb{R}^k(F)$.

$$\lambda_0 x_0 + \lambda_1 x_1 + \dots + \lambda_k x_k = 0$$

$$\lambda_0 < \alpha < \left[\lambda_0 \varphi_0(u) + \dots + \lambda_k \varphi_k(u) \right] \quad \left[\begin{array}{l} u \rightarrow \mu u \\ \varphi_i(u) \rightarrow \mu^i \varphi_i(u) \end{array} \right]$$

$$\Rightarrow \lambda_0 \varphi_0(u) + \dots + \lambda_k \varphi_k(u) = 0 \quad \forall u \in V \quad \mu \in \mathbb{R}$$

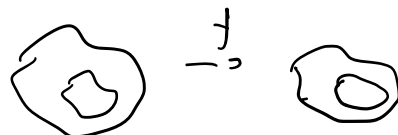
$$\Rightarrow \lambda_0 < 0$$

$$\varphi(u) = - \sum_{i=1}^k \frac{\lambda_i}{\lambda_0} \varphi_i(u)$$

נ"ל של φ הוא פונקציה ליניארית. $\exists V \in \mathcal{L}(V)$ (בדבריה)

$$|\varphi(v)| < 1 \quad \forall v \in V$$

$$\text{נס' } V = \{x_i \in V : |\langle f, x_i \rangle| < \delta, i=1, \dots, k\}$$



$$\langle f, x_i \rangle > 0, \quad \forall i=1, \dots, k. \Rightarrow \varphi(x_i) = 0$$

$$|\langle f, x_i \rangle| < \delta \Rightarrow |\langle \frac{f}{\delta}, x_i \rangle| < 1. \Rightarrow \frac{f}{\delta} \in V. \Rightarrow |\varphi(\frac{f}{\delta})| < 1 \Rightarrow \varphi(f) < \delta$$

$$\varphi = \sum_{i=1}^k \lambda_i x_i \in V$$

$$\varphi(f) = \langle f, x \rangle \quad \forall f \in V. \quad \text{אז } \varphi \in \mathcal{L}(V)$$

(Banach-Alaoglu) E 赋范, $(\overline{B_{E^*}}, \sigma(Z^*, Z))$ 紧. e^*

E^* 的闭单位球是弱*紧的.

$\rho f: E^* \subset \mathbb{K}^Z, \quad \forall f \in E^*, \quad \underbrace{\{f(x)\}}_{x \in Z} \in \mathbb{K}^Z.$

$(Z^*, \sigma(Z^*, Z))$ 可积或是 \mathbb{K}^Z 上的序列拓扑.

(x_i^*) 是 E^* 中序列. $x_i^* \rightarrow x^*$. 收敛的序列 $\Leftrightarrow x_i^* \rightarrow x^*$ in \mathbb{K}^Z .

$\Leftrightarrow \forall x \in Z. \langle x_i^*, x \rangle \rightarrow \langle x^*, x \rangle.$

$\Leftrightarrow x_i^* \rightarrow x^*$ in $\mathbb{K}^Z.$

$(Z^*, \sigma(Z^*, Z))$ 可积 \mathbb{K}^Z 上的序列拓扑诱导.

$\overline{B_{E^*}}, \sigma(Z^*, Z) |_{\overline{B_{E^*}}}$, 紧 \mathbb{K}^Z 在 $\overline{B_{E^*}}$ 上闭包到

$\forall f \in Z^*, f \mapsto f|_{\overline{B_Z}}$.

$\overline{B_{E^*}} = \left\{ x^* \in E^* \mid |x^*(x)| \leq 1, \forall x \in \overline{B_Z} \right\}$

则 $\overline{B_{E^*}}$ 中元素. 在此意义下收敛于 $B_{\mathbb{K}} = \mathbb{K}$.

$\forall f \in \overline{B_{E^*}}$. f 收敛于 $f|_{\overline{B_Z}}: \overline{B_Z} \rightarrow \mathbb{K}$. $\overline{B_{E^*}}$ 紧 \mathbb{K} $\overline{B_Z} = \emptyset$ 集.

\mathbb{K} 紧. $\mathbb{K}^{\overline{B_Z}}$ 紧. $\overline{B_{E^*}}$ 紧 $\Rightarrow \overline{B_{E^*}}$ 紧. (闭-紧)

$\forall \varphi \in \overline{B_{E^*}}$. φ 是 $\overline{B_Z}$ 上连续线性映射, $\forall x, y \in \overline{B_Z}, \alpha, \beta \in \mathbb{K}$. \mathbb{K} .

$\alpha x + \beta y \in \overline{B_Z}, \quad \varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y)$

若 $\varphi: \overline{B_Z} \rightarrow \mathbb{K}$ 线性. 则 φ 收敛到 $\overline{B_{E^*}}$ 中元素.

$\forall x \in Z. \exists \lambda \neq 0, \text{ s.t. } \frac{x}{\lambda} \in \overline{B_Z}. \quad \varphi(x) = \lambda \varphi\left(\frac{x}{\lambda}\right)$

$\varphi \in \overline{B_{E^*}}$,

设 $\varphi \in \mathbb{K}^{\overline{B_Z}} \setminus \overline{B_{E^*}}$. 则有 $x_1, x_2 \in \overline{B_Z}, \alpha, \beta \in \mathbb{K}. \alpha x_1 + \beta x_2 \in \overline{B_Z}$

$\varphi(\alpha x_1 + \beta x_2) \neq \alpha \varphi(x_1) + \beta \varphi(x_2)$

设 $x_3 = \alpha x_1 + \beta x_2, \in \bar{B}_0$.

$$V = V(\gamma; x_1, x_2, x_3, \varepsilon) = \{ f \in F^{\bar{B}_0} : |(f-\gamma)(x_i)| < \varepsilon, i=1,2,3 \}$$

$$|f(x_3) - \alpha f(x_1) - \beta f(x_2)| \geq |\gamma(x_3) - \alpha \gamma(x_1) - \beta \gamma(x_2)| > 0$$

$$- |f-\gamma(x_3)| - (\alpha |f-\gamma(x_1)| + |\beta| |f-\gamma(x_2)|)$$

> 0 . 矛盾

$\Rightarrow V \subset F^{\bar{B}_0} \setminus \bar{B}_0^*$, $\Rightarrow \bar{B}_0^*$ 在 $F^{\bar{B}_0}$ 闭 $\Rightarrow \bar{B}_0^*$ 闭. (紧致-拓扑)

$\Rightarrow \bar{B}_0^*$ 闭

(Goldstein). Z 闭. B_Z 在 \bar{B}_Z^{**} 中为 * 和 \bar{B}_Z .

$$\bar{B}_Z^{w*} = \bar{B}_Z^{**} \quad (Z^{**}, \sigma(Z^{**}, Z^*))$$

pf: ① $B_Z \subset \bar{B}_Z^{**}$, $x \in B_Z, \hat{x} \in B_Z^{**}, x \rightarrow \hat{x}$.

证 $\bar{B}_Z^{w*} \subset \bar{B}_Z^{**}$ 需证 \bar{B}_Z^{**} 是 Z 闭.

$\forall \|\varphi\|_{Z^{**}} > 1$. 找 $\varphi \in B_Z^{**}$ 使 $\|\varphi\|_{Z^*} > 1$. 故 $V \subset (\bar{B}_Z^{**})^c$.

$\exists f \in Z^*, \|\varphi\|_{Z^*} \leq 1, |\varphi(f)| > 1$.

$B(\varphi, \varepsilon) = \{ \gamma \in Z^{**} : |(\varphi - \gamma)(f)| < \varepsilon \}$. $\varphi \in B_Z^{**}$ 矛盾

$\varepsilon > 0, |\varphi(f)| > 1, \Rightarrow \varphi \notin \bar{B}_Z^{**} \Rightarrow B(\varphi, \varepsilon) \subset (\bar{B}_Z^{**})^c$.

② $\bar{B}_Z^{w*} = \bar{B}_Z^{**}$. 证 $\exists \varphi \in \bar{B}_Z^{**} \setminus \bar{B}_Z^{w*}, (Z^{**}, \sigma(Z^{**}, Z^*))$

$\{\varphi\}$ 闭. \bar{B}_Z^{w*} 闭

Z .
(拓扑)

A, B 闭. $A \cup B$ 闭

根据闭子集性质. 不在

$\exists f \in Z^*, |f(A)| < \delta < |f(B)|$

$$f \in (Z^{**}, \sigma(Z^{**}, Z^*))^* = E^* \quad (= Z^*)$$

$f: (Z^{**}, \sigma(Z^{**}, Z^*)) \rightarrow (K, \tau_0)$ 上之连续线性映射.

s.t. $\langle f, \varphi \rangle > 1$. \square $\langle f, \bar{B}_2^{W^*} \rangle \leq 1$

$B_2 \subset B_2^{W^*}$

$\|f\|_{Z^*} = \sup_{x \in B_2} |\langle f, x \rangle| = \sup_{x \in B_2} |\langle \bar{x}, f \rangle| \leq 1$

$\|\varphi\|_{Z^{**}} \geq |\langle f, \varphi \rangle| > 1$ \square

Z^* 范数 $\tau: Z \rightarrow Z^*$ $x \mapsto \bar{x}$, 其中 $\bar{x}(f) = f(x)$, $\forall f \in Z^*$.

则 τ 是 Z 到 Z^* 的等距同构.

$\|\bar{x}\|_{Z^*} = \|x\|_Z$

$\|x\|_Z = \sup_{\|f\|_{Z^*} \leq 1} |\langle f, x \rangle| = \sup_{\|f\|_{Z^*} \leq 1} |\langle \bar{x}, f \rangle| = \|\bar{x}\|_{Z^*}$

Z 范数.

$f(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 = 0$

Z^* 范数: $\tau(x_1) = \tau(x_2)$, $\forall f \in Z^*$, $f(\tau(x_1)) = f(\tau(x_2)) \Rightarrow x_1 = x_2$.

$\forall y \neq 0$, $\exists f \in Z^*$, $f(y) \neq 0$.

Def. Z 范数. τ 是 Z 到 Z^* 的等距同构.

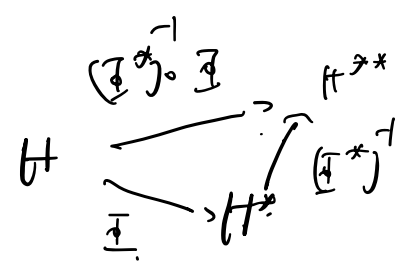
eg. Hilbert 空间 H .

$H \cong H^* \cong H^{**}$

\Rightarrow Riesz 定理, $\exists \bar{J}: H \rightarrow H^*$ 等距同构.

$\bar{J}^*: H^{**} \rightarrow H^*$ 同构.

$\underbrace{(\bar{J}^*)^{-1} \circ \bar{J}}_{= \tau}: H \rightarrow H^{**}$ 同构



(2, M, R)

egz. (\mathbb{R}, μ) simple measure space. (\mathbb{R}, μ) $L_p(\mathbb{R}, \mu)$ \mathbb{R}

$1 < p < \infty$ $(L_p)^* \cong L_q$ $(L_p)^{**} \cong (L_q)^* \cong L_p$

$\forall g \in L_q \quad \varphi_g(f) = \int_{\Omega} fg$ $\forall f \in L_p \quad \varphi_g \in (L_p)^*$

$\| \varphi_g \| = \| g \|_{L_q}$ $L_q \subset (L_p)^*$

for $(L_p)^* \subset L_q$ Radon-Nikodym

$\forall \varphi \in (L_p)^*$ $\exists h \in L_q$ $\varphi(z) = \int_{\Omega} h(z) d\mu$ $\varphi \ll \mu$

$\exists g \in L^1(\mu)$ so $\varphi(z) = \int_{\Omega} g d\mu$

with $g = \sum_{i=1}^n a_i \chi_{z_i}$

$\varphi(h) = \sum_{i=1}^n a_i \varphi(z_i) = \sum_{i=1}^n \int_{z_i} a_i g d\mu = \int_{\sum_{i=1}^n a_i \chi_{z_i}} g d\mu$
 $= \int u g d\mu$

$\Rightarrow \varphi(f) = \int fg d\mu$ $\forall f \in L_p$

Z F normed space. $u \in B(Z, F)$. $\exists! u^* \in B(F^*, Z^*)$ s.t.

for $\forall f^* \in F^*$, $x \in Z$. \forall

$\langle u^*(f^*), x \rangle = \langle f^*, u(x) \rangle$

particular, u normed, $\Rightarrow u^*$ normed

Thm. Banach space Z reflexive $\Leftrightarrow Z^*$ reflexive

pf: " \Rightarrow " Z reflexive $Z = Z^{**} \Rightarrow Z^* = (Z^*)^{**}$

$\tau: Z \rightarrow Z^{**}$ isometric

$\tau^*: Z^{***} \rightarrow Z^*$ isometric $\Rightarrow Z^* = (Z^*)^{**}$

" \Leftarrow " Z^* reflexive $Z^* = Z^{***}$, \exists normed Z reflexive $\Rightarrow Z = Z^{**}$

egz. for $\forall f \in (Z^{**})^*$, $f|_Z = 0 \Rightarrow f = 0$

$$f \in Z^{***}, \dots f \in Z^* \quad f|_Z = 0 \Rightarrow f = 0$$

$$(Z, \mu) \text{ } \sigma\text{-meas.} \quad (L^1(\mu))^* = L^\infty(\mu)$$

Thm. 对任意内积 \$Z, F\$ 内积. \$Z\$ 内积 \$\Leftrightarrow F\$ 内积

pf: \$F\$ 内积 \$\Rightarrow Z\$ 内积

\$u: Z \to F\$ 内积, \$\exists u^*: F^* \to Z^*\$ 内积 \$\exists u^{**}: Z^{**} \to F^{**}\$ 内积

$$\text{Claim. } u^{**}|_Z = u \quad \forall x \in Z \quad \frac{u^{**}(\hat{x})}{\|x\|} = \widehat{u(x)}$$

$$\forall f^* \in F^* \quad \langle u^{**}(\hat{x}), f^* \rangle = \langle \widehat{u(x)}, f^* \rangle$$

$$\begin{aligned} \text{LHS} &= \langle \hat{x}, u^*(f^*) \rangle = \langle u^*(f^*), x \rangle = \langle f^*, u(x) \rangle \\ &= \langle \widehat{u(x)}, f^* \rangle \quad \Rightarrow \quad u^{**}(\hat{x}) = \widehat{u(x)} \end{aligned}$$

$$F \text{ 内积 } F = F^{**} \quad u^{**}|_Z = u: Z \to F \text{ 内积}$$

$$u^{**}: Z^{**} \to F^{**} \text{ 内积} \quad \Rightarrow Z = Z^{**}$$

eg. \$L^1, L^\infty\$ 不是内积.

$$Z \text{ 内积, } \|x\| = \sup_{\|f\|_{Z^*} \leq 1} |\langle f, x \rangle| \quad \text{上确界. 对. } \|x\| = |\langle f, x \rangle|$$

$$\forall \varphi \in Z^* \quad \|\varphi\| = \sup_{\|x\| \leq 1} |\langle \varphi, x \rangle| = \sup_{\|x\| \leq 1} |\langle \varphi, x \rangle|$$

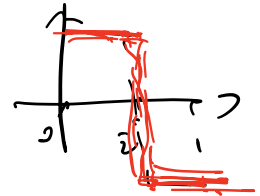
$$\exists x \in Z, \|x\| \leq 1 \quad \text{s.t. } \|\varphi\| = |\langle \varphi, x \rangle|$$

eg 1. $C([0,1])$ 不是 Hilbert

$\varphi = \int_0^{\cdot} f - \int_{\cdot}^1 f$ (线性. 连续. $f_n \rightarrow f \Rightarrow \varphi(f_n) \rightarrow \varphi(f) \Rightarrow \varphi \in (C([0,1]))^*$)

$\forall \|f\|_{L^\infty} \leq 1, \sup_{\|f\|_{L^\infty} \leq 1} |\varphi(f)|$

$|\varphi(f)| = \left| \int_0^{\cdot} f dx - \int_{\cdot}^1 f dx \right|$



$\leq \int_0^{\cdot} \|f\|_{L^\infty} dx + \int_{\cdot}^1 \|f\|_{L^\infty} dx \leq 1$

$\Rightarrow \|\varphi\| \leq 1$

$\|\varphi\| \geq 1 - \varepsilon, \forall \varepsilon > 0 \Rightarrow \|\varphi\| = 1$

$\langle \varphi, f \rangle = 1, \|f\|_{L^\infty} = 1, a.e. x \in [0,1] \quad f \equiv 1, f \equiv -1$

$\langle \varphi, f \rangle = 0$ 其他

eg 2. $L^1([0,1])$ 不是 Hilbert, $(L^1([0,1]))^* = L^\infty([0,1]) \Rightarrow L^1([0,1])$ 不是 Hilbert

$\varphi(f) = \int_0^1 f(x) x dx$ (x circled) φ 连续. $f_n \xrightarrow{L^1} f \Rightarrow \varphi(f_n) \rightarrow \varphi(f)$

$\forall \|f\|_{L^1} \leq 1, |\varphi(f)| \leq \|f\|_{L^1} \|x\|_{L^\infty} = \|f\|_{L^1} \leq 1 \Rightarrow \|\varphi\| \leq 1$

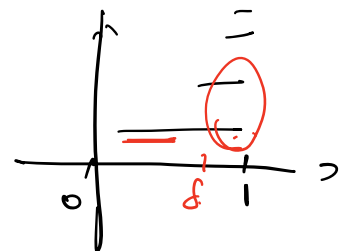
$\|\varphi\| \geq 1 - \varepsilon, f_n = \begin{cases} 0 & [0, 1-\frac{1}{n}] \\ n & [1-\frac{1}{n}, 1] \end{cases}$

$\varphi(f_n) = \int_{1-\frac{1}{n}}^1 n x dx = \frac{n}{2} (1 - (1-\frac{1}{n})^2) = \frac{1}{2} (2 - \frac{1}{n}) = 1 - \frac{1}{2n}$

$\Rightarrow \|\varphi\| \geq 1 - \varepsilon \Rightarrow \|\varphi\| = 1$

$\exists f \in L^1([0,1]), \|f\|_{L^1} \leq 1, \langle \varphi, f \rangle = 1$

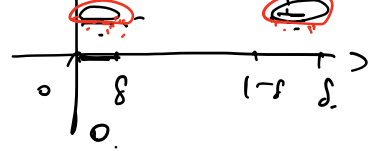
例 $f \in L^1([0,1]) \Rightarrow 0, a.e.$ $\forall 0 < \delta < 1$



其他 $\forall 0 < \delta < 1, \exists f \in L^1$

$\tilde{f} = \begin{cases} 0 & (0, \delta) \\ 1 & (\delta, 1) \end{cases}$

$$\int_{\delta}^{1-\delta} f(x) + f(x-u-\delta) \omega(\delta, \nu)$$



$$\|f\|_{L^1(\omega, \nu)} = \|f\|_{L^1(\omega, \nu)} \quad \text{2b}^2 \text{ipm} \text{is } f_{2-\nu}$$

$$\int_0^1 |f| = \int_{\delta}^{1-\delta} |f| + \int_{1-\delta}^1 |f(x) + f(x-u-\delta)| \leq \int_0^1 |f(x)| \leq 1$$

$$p(\hat{f}) > p(f) = 1 \quad \text{for } \mu \leq 1$$

for p -norm in (X, μ, ν) , $1 < p < \infty$ $(L^p(\mu))^* = L^q(\nu)$

μ -measure in ν ,

$$p=1, \quad \mu \text{ is } \nu\text{-type.} \quad (L^1(\mu))^* = L^\infty(\nu)$$

Thm. Z is K . Z is $K \iff \bar{B}_Z$ is w -type.

$$\Rightarrow " \quad Z \text{ is } K \quad Z = Z^{**}, \quad \bar{B}_Z = \bar{B}_{Z^{**}} \quad \bar{B}_{Z^{**}} \text{ is } w\text{-type } \iff \bar{B}_Z \text{ is } w\text{-type.}$$

$$\Leftarrow " \quad \bar{B}_Z \text{ is } w\text{-type.} \quad \bar{B}_Z \text{ is in } [Z^{**}, \sigma(Z^{**}, Z^*)] \text{ is } w\text{-type.}$$

$$\Rightarrow \bar{B}_Z \text{ is in } [Z^{**}, \sigma(Z^{**}, Z^*)] \text{ is } w\text{-type.}$$

$$\Rightarrow \bar{B}_Z = \bar{B}_Z^{w*} \quad \text{Goldstein.} \quad \bar{B}_{Z^{**}} \iff Z \text{ is } K$$

\bar{B}_{Z^*} is w -type.

$$\text{if } \bar{B}_{Z^*} \text{ is } w\text{-type} \iff \dim Z^* < \infty \iff \underline{\dim Z} < +\infty$$

$$\text{if } \bar{B}_{Z^*} \text{ is } w\text{-type} \iff Z \text{ is } K$$

$$\text{if } \bar{B}_{Z^*} \text{ is } w\text{-type}$$

自反空间.

$\Omega \subset \mathbb{R}^n$, $L^p(\Omega)$ 自反, $1 < p < \infty$.

$L^1(\Omega)$, $L^\infty(\Omega)$ 不是自反的.

$$(L^p)^* \cong L^q, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{对于 } 1 < p < \infty. \quad (L^p)^{**} \cong (L^q)^* \cong L^p.$$

$$(L^1)^* \cong L^\infty.$$

prop. $W^{k,p}(\Omega)$, $1 < p < \infty$, 自反

$$u \in W^{k,p}(\Omega) \quad (u, Du, D^2u, \dots, D^k u) \in (L^p)^N, \quad N = N(k)$$

$$\varphi : W^{k,p}(\Omega) \rightarrow \varphi(W^{k,p}(\Omega)) \subset (L^p)^N \quad (a^0, a^1, \dots, a^k), \quad \sum_{i=0}^k \|a^i\|_{L^p}$$

$$u \mapsto (u, Du, \dots, D^k u)$$

φ 同构, 单射, 22 射, φ 连续, $\|\varphi(u)\| = \|u\|$

φ^{-1} 连续: \Leftarrow 闭映射定理, $\varphi(W^{k,p}(\Omega))$ Banach?

$$\forall \varphi(W^{k,p}(\Omega)) \ni (u_m, Du_m, \dots, D^k u_m) \rightarrow (v, v^1, \dots, v^k)$$

$$\exists v \in W^{k,p}(\Omega) \quad (v, v^1, \dots, v^k) \in \varphi(W^{k,p}(\Omega)) \quad \text{Div } v = v^i$$

$$\int v^i \phi = \lim_m \int D^i u_m \phi = - \lim_m \int u_m D^i \phi = - \int v D^i \phi$$

$\Rightarrow v \in W^{k,p}(\Omega)$. $\text{Div } v = v^i$. $\Rightarrow \varphi(W^{k,p}(\Omega))$ 闭. $\Rightarrow \varphi$ 同构

$(L^p(\Omega))^N$ 自反, 由 Pettis 定理.

(Pettis). 自反空间 Z 之闭子空间 M 也自反.

pf: Banach 空间 Z 自反 $\Leftrightarrow B_Z$ 是弱闭的

$$\text{Banach-Alaoglu Thm. } Z^{**}, B_{Z^{**}} \text{ 在 } \sigma(Z^{**}, Z^*) \text{ 下弱闭.}$$

$$\frac{1}{Z} \quad \frac{1}{B_Z} \quad \frac{1}{\sigma(Z, Z^*)}$$

$$(\underline{z}, \underline{\sigma(z, z^*)}) \cong (\underline{z^*}, \underline{\sigma(z^*, z^*)})$$

$$\varphi: x \mapsto \hat{x}$$

$$x_n \rightarrow x, \quad \forall f \in \mathcal{Z}^*, \quad f(x_n) \rightarrow f(x)$$

$$\hat{x}_n \xrightarrow{*} \hat{x} \iff \forall f \in \mathcal{Z}^* \quad f(x_n) \rightarrow f(x)$$

记 $(B_M, \sigma(M, M^*))$ F^{nd} \Rightarrow M 是 K

$$\textcircled{1} \sigma(z, z^*)|_M \quad \textcircled{2} \sigma(M, M^*) \text{ 是 } F^{\text{nd}}$$

$$\textcircled{1} \Rightarrow \textcircled{2}$$

对于 $\textcircled{1}$ 中 F^{nd} 在 F 中是 $\{x \mid |f_i(x)| < \varepsilon, i=1, \dots, n\} \cap M, f_i \in \mathcal{Z}^*$

$\textcircled{2}$ 中 F^{nd} 在 F 中是 $\{x \in M \mid |g_j(x)| < \delta, j=1, \dots, n\}, g_j \in M^*$

$f_i \times 0$ 中 F^{nd} , g_j 和 $f_i|_M \in M^*$, δ 和 ε 是 F 中 ε . $\tilde{g}_j|_M = g_j$

$f_i \times 0$ 中 F^{nd} , $g_j \in M^*$, $\underline{H} = \underline{B}$, $\tilde{g}_j, \|\tilde{g}_j\| (= \|g_j\|), \tilde{g}_j \in \mathcal{Z}^*$

把 f_i 和 g_j 看成 F 中 ε 和 δ 的 F^{nd} .

$$\Leftrightarrow (B_M, \sigma(z, z^*)|_M) F^{\text{nd}}$$

$$F \text{ 是 } K \Rightarrow (B_Z, \sigma(z, z^*)) F^{\text{nd}}$$

$$M \text{ 是 } F \text{ 中 } \mathbb{R} \text{ 的闭子集, } B_M = M \cap B_Z \text{ 在 } \sigma(z, z^*)|_M F^{\text{nd}} \Rightarrow B_M \text{ 在 } \sigma(M, M^*) F^{\text{nd}}$$

可分空间.

Banach-Algebra T 上 B_Z^* 是 T^* , 但不是 T 中 T^* 的 F^{nd}

eg. $Z = \ell_\infty$, B_Z^* 是 T^* , 但不是 T 中 T^* 的 F^{nd}

$$\text{对于 } \tilde{x}^{(n)} \in B_Z^*, \quad x^{(n)} = (0, \dots, 0, \overset{\text{第 } n \text{ 个}}{2}, 0, \dots, 0) \quad \|x^{(n)}\| \leq 1$$

$$\text{对于 } x^{(n)} \xrightarrow{B_Z^*} x = (x_1, \dots, x_n, \dots), \text{ 对于 } \forall a = (a_i) \in \ell_\infty$$

$$\langle x^{(n)}, a \rangle \rightarrow \langle x, a \rangle \quad n \rightarrow \infty$$

$$\langle x^{(n)}, a \rangle = a_n, \quad \langle x, a \rangle = \sum_n x_n a_n.$$

$$a_n \rightarrow \sum_n x_n a_n, \quad \forall x = (a_n) \in l_\infty \text{ 或 } l_1,$$

$$\forall a = (1, 0, \dots, 0) \quad a_n \rightarrow 0, \quad \sum_n x_n a_n = x_1 \Rightarrow x_1 = 0;$$

$$\text{同法 } x_1 = x_2 = \dots = 0 \Rightarrow x = 0$$

$$\forall a = (1, \dots, 1, \dots) \quad a_n \rightarrow 1, \quad \sum_n x_n a_n = 0 \quad (x=0) \text{ 矛盾.}$$

$\Rightarrow x^{(n)}$ 不是弱*收敛的, 即它没有弱*收敛子列.

范数收敛, $\| \cdot \| \Rightarrow$ 收敛, 弱*收敛, $\| \cdot \|_*$ 收敛, $\| \cdot \|_*$ 收敛.

Remark: $\dim \mathcal{E} = +\infty$.

$(\mathcal{E}, \sigma(\mathcal{E}, \mathcal{E}^*))$ 不是 \mathcal{E} 的 σ -范数, (Brezis).

$(\mathcal{E}, \sigma(\mathcal{E}, \mathcal{E}^*))|_{\mathcal{E}}$ 不是 \mathcal{E} 的 σ -范数, $(\mathcal{E}^*, \sigma(\mathcal{E}^*, \mathcal{E}))|_{\mathcal{E}^*}$ 是 \mathcal{E}^* 的 σ -范数.

L^∞ 闭单位球, 不是弱*收敛的 \Leftarrow Banach-Alaoglu + L^2 可分.

$(L^\infty)^*$ 闭单位球不是弱*收敛的 $\Leftarrow L^\infty$ 不可分.

Def. 弱*收敛子列的存在性定理.

eg. $\Omega \subset \mathbb{R}^n, L^p(\Omega), 1 \leq p < \infty$ 可分, $L^\infty(\Omega)$ 不可分, $(L^\infty)^*$ 不可分.

$L^p(\Omega), 1 \leq p < \infty$

在 Ω 中任取一个点 $x \in \mathbb{Q}^n \cap \Omega$ 处, $B(x, \frac{1}{n}), n \geq 1$.

$B(x, \frac{1}{n}), n \in \mathbb{N}, x \in \mathbb{Q}^n \cap \Omega$, 可分.

考虑它内部有收敛子列 \Rightarrow 弱*收敛子列 \Rightarrow 弱*收敛子列 \Rightarrow \mathcal{E} .

$$\sum_{i=1}^m a_i \chi_{B(x_i, \frac{1}{n_i})}, \quad a_i \in \mathbb{Q}, \quad x_i \in \mathbb{Q}^n \cap \Omega, \quad n_i \in \mathbb{N}.$$

$\forall f \in L^p(\Omega) \exists \psi_n$ 紧支 $\|f - \psi_n\|_{L^p} \leq \epsilon$

$\rho = \sum_{j=1}^k b_j \chi_{B_j}$, B_j 互不相交 \Leftarrow 开集覆盖 \Leftarrow 紧集性质

$\exists \psi \in \mathcal{S}$. $\|\psi - \psi_n\|_{L^p} \leq \epsilon \Rightarrow \mathcal{S}$ 在 L^p 中稠密

l^∞ . $\mathcal{S} = \{ (x_n) \mid x_n = 0 \text{ 或 } \pm 1 \}$. $\forall (a_n), (b_n) \in \mathcal{S}$. $(a_n) \neq (b_n)$,

$\|(a_n) - (b_n)\|_{l^\infty} = 1$. \mathcal{S} 不稠密 $\Rightarrow l^\infty$ 不紧



$C(K)$ 紧

$L^\infty(\Omega)$. 任取 $x \in \Omega$. $B(x, \delta) \subset \Omega$. 对 $\forall \delta$. $\exists B(x, \frac{\delta}{R})$. $R \geq 1$.

$\forall R_1, R_2$. $R_1, R_2 \geq 1$. $\|\chi_{B(x, \frac{\delta}{R_1})} - \chi_{B(x, \frac{\delta}{R_2})}\|_{L^\infty} = 1$.

因为 $\{\chi_{B(x, \frac{\delta}{R})}, R \geq 1\}$ 不收敛 $\Rightarrow L^\infty(\Omega)$ 不紧

Prop. E 紧子集是闭集, $F \subset E$ 闭子集, 则 F 紧

pf. E 紧 $\Rightarrow E$ 闭且有限. $D \cap F$ 紧. $D \cap F \subset F$ 闭 $\Rightarrow F$ 紧

(u_n) 是 E 中紧子集, $r_n \rightarrow 0$

$a_n \in B(u_n, r_n) \cap F$. 则 $a_n \in E$,

且在 F 中稠密



$x_0 \in F$, $x_0 \in D$

$x_0 \in D \cap F$ $x_0 \in B(x_0, r_n) \cap F$

Thm. E Banach 空间, E^* 紧, 则 E 紧. (反之不真, L^1 紧, L^∞ 不紧)

pf: (f_n) 是 E^* 中紧子集, $\exists x_n, \|x_n\| = 1$,

$\langle f_n, x_n \rangle \geq \frac{1}{2} \|f_n\|$. 选 L_0 为 $(x_n)_{n \geq 1}$ 的弱收敛子列

找子列, L_0 在 $L = \text{span}\{x_1, \dots, x_n, \dots\}$ 中稠密

只需证明 L 在 E 中稠密, 从而 $\forall f \in E^*$, $f|_L = 0 \Rightarrow f = 0$?

$$f \neq f. \exists f_n. \text{ s.t. } \|f - f_n\| < \varepsilon$$

$$\frac{1}{2} \|f_n\| \leq \langle f_n, x_n \rangle = \langle f_n - f, x_n \rangle \leq \|f_n - f\| < \varepsilon$$

$$\Rightarrow \|f_n\| \leq 2\varepsilon. \Rightarrow \|f\| \leq \|f_n\| + \|f - f_n\| < 3\varepsilon. \Rightarrow f = 0.$$

$$\Rightarrow L^2 \text{ 不可分. } \Rightarrow L_0^2 \text{ 不可分. } L_0^2 \text{ 不可分 } \Rightarrow \text{不可分}$$

这不行 \Rightarrow 这不行

Thm. 不可分 Banach 空间, $(B_{Z^*}, \sigma(Z^*, Z) |_{B_{Z^*}})$ 不可分

反之. 若 $(B_{Z^*}, \sigma(Z^*, Z) |_{B_{Z^*}})$ 不可分 $\Rightarrow Z$ 不可分.

pf: " \Rightarrow " 令 (x_n) 是 Z 中两两正交的 $\forall f \in Z^* \quad |\langle f, x_n \rangle| \leq \|f\| \|x_n\|$
 $[f] = \sum \frac{1}{2^n} |\langle f, x_n \rangle|$ (x_n) 是 Z 中两两正交的 $\leq \|f\|$

$[f]$ 是 Z^* 上之范数, 且 $[f] \leq \|f\|$, $d(f, g) = [f - g]$.

Claim. 范数 $[f]$ 在 B_{Z^*} 上之拓扑与 $(B_{Z^*}, \sigma(Z^*, Z) |_{B_{Z^*}})$ 一致.

Let $f_0 \in B_{Z^*}$, $\forall \varepsilon > 0$ 存在 $\delta > 0$ 使得 $r = \delta$ 时,

$$V = \{ f \in B_{Z^*} \mid |\langle f - f_0, y_i \rangle| < \varepsilon, y_i \in Z, \forall i = 1, \dots, k \}$$

$$\text{且使得 } U = \{ f \in B_{Z^*} \mid d(f, f_0) < r \}, r \text{ 足够小, s.t. } U \subset V.$$

不妨假设 $\|y_i\| < 1, \forall i = 1, \dots, k$, 则对于 $i = 1, \dots, k$, 存在 $x_{ni} \in Z$.

$$\|y_i - x_{ni}\| < \frac{\varepsilon}{4}, \text{ 选择 } r \text{ 足够小. } r < \frac{\varepsilon}{2^{k+1}}$$

$$d(f, f_0) < r \Rightarrow \frac{1}{2^{ni}} |\langle f - f_0, x_{ni} \rangle| < r, \forall i = 1, \dots, k.$$

$$|\langle f - f_0, y_i \rangle| \leq |\langle f - f_0, y_i - x_{ni} \rangle| + |\langle f - f_0, x_{ni} \rangle| < \frac{\varepsilon}{2} \quad (\leq \varepsilon)$$

$$\text{且使得 } f \in \text{球 } V = \{ f \in B_{Z^*} \mid |\langle f - f_0, y_i \rangle| < \varepsilon, \forall i = 1, \dots, k \} \subset U.$$

取 $x_{ni} = y_i$.

$$d(f, f_0) = \sum_n \frac{1}{2^n} |\langle f - f_0, x_n \rangle| = \sum_{n=1}^k \frac{1}{2^n} |\langle f - f_0, x_i \rangle| + \sum_{n=k+1}^{\infty} \frac{1}{2^n} |\langle f - f_0, x_n \rangle|$$

$$\leq \varepsilon \cdot 1 + \frac{1}{2^k} |\langle f - f_0, x_i \rangle| \leq \varepsilon$$

$\exists \varepsilon > 0, \exists x \in X, d(f, 0) < \varepsilon, \|f(x)\| < \varepsilon \|x\|$ ($\|f\| + \|f_0\|$) $\frac{\|x\|}{\varepsilon}$

Remark. $[f] = \sum \frac{1}{2^k} | \langle f, x_k \rangle |$, $(x_k) \subset B_{Z^*}$
 $\forall f \in Z^*$, $[f]$ 是 Z^* 上之范数, $(Z^*, [f]) \rightarrow$ 拓扑.
 $(Z^*, [f]) \neq (Z^*, \sigma(Z^*, Z))$
 $(B_{Z^*}, [f]) = (B_{Z^*}, \sigma(Z^*, Z)|_{B_{Z^*}})$

" \Leftarrow " $(B_{Z^*}, \sigma(Z^*, Z)|_{B_{Z^*}})$ 是 separable $\Rightarrow Z$ 可分.

$U_n = \{ f \in B_{Z^*} \mid d(f, 0) < \frac{1}{n} \}$

选择 v_n 是 0 在 Z^* 上之邻域, 故 $V_n \subset U_n$,

$V_n = \{ f \in B_{Z^*} \mid | \langle f, x \rangle | < \varepsilon_n, \forall x \in I_n \}$ I_n 稠密.

$D = \bigcup I_n \subset Z$.

U_n 的闭包为 Z^* 中所有范数 $< \frac{1}{n}$ 之函数 L_n .

$Z^* \supset L_n \supset L_n \text{ 在 } Z^* \text{ 中}$

L_n 在 Z^* 中稠密.

$\forall f \in Z^*, f|_{L_n} = 0 \Rightarrow \langle f, x \rangle = 0, \forall x \in D \Rightarrow f \in V_n \Rightarrow f \in U_n \Rightarrow f = 0$

Thm. Z Banach, Z^* 可分 $\Rightarrow B_{Z^*}$ 在 $\sigma(Z^*, Z)$ 下可分

反之. 若 B_{Z^*} 在 $\sigma(Z^*, Z)$ 下可分 $\Rightarrow Z^*$ 可分.

L^2 L^2

Cor. Z 可分 Banach 空间, $Z|_{Z^*}$ 上之范数是弱*范数.

特别地, (f_n) 是 Z^* 中之序列, $Z|_{Z^*}$ 上之范数是弱*范数.

pf: Banach-Algebra Th. $\Rightarrow (B_{\mathbb{C}^*}, \sigma(\tau^*, \tau))|_{B_{\mathbb{C}^*}}$ 116.

$$\Downarrow \tau \text{ 可分}$$

$$\Downarrow \text{Th 116}$$

116 42 \Leftrightarrow 2116 42

$$\Downarrow$$

$$(B_{\mathbb{C}^*}, \sigma(\tau^*, \tau)|_{B_{\mathbb{C}^*}}) \text{ 可分}$$

(Th 116*) \mathbb{R} -Banach 空间, 有界闭集的 χ 子集 $= \{1\}$.

pf. τ 可分, 它是闭的, \Rightarrow 可分 116 42,

$\{x_n\} \subset \tau$ 有界闭集, $\|x_n\| \leq 1$. 其 $\bar{\mu}$ 是 $\{x_n\}$ 的 τ -闭包, $\mu = \bar{\mu}$, 是闭子空间. 可分, 可分, $(B_{\mu}, \sigma(\tau, \tau^*)|_{B_{\mu}})$ 可分.

(Remark, Remark τ 可分 $\Leftrightarrow \tau^*$ 可分 $\Leftrightarrow \tau$ 可分)

\Leftrightarrow 可分 $\Rightarrow \tau$ 可分 + 可分. τ^{**} 可分 $\Rightarrow \tau^*$ 可分 $\Rightarrow \tau$ 可分 + 可分.

μ^* 可分, 可分 $(B_{\mu}, \sigma(\tau, \tau^*)|_{B_{\mu}})$ 可分. (116 42)

$\Rightarrow B_{\mu}$ 是可分 116.

$$\underline{W^{-m, p}(\Omega)} = \left(\underline{W_0^{-m, p}(\Omega)} \right)^*$$

$$\left(\underline{W_0^{-m, p}(\Omega)} \right)^* + \mathbb{R}k$$

$$H^1 = L^2 \subset H^{-1}$$

$$\underline{W_0^{-m, p}(\Omega)} \subset W^{-m, p}(\Omega)$$

Thm (Zoberlein-Simuliam)

τ Banach 空间, τ 可分 116 42 \Leftrightarrow 可分 116 42.

\Rightarrow (*) $\mathbb{R}k \Leftrightarrow (B_{\tau}, \sigma(\tau, \tau^*)|_{B_{\tau}})$ 可分 \Leftrightarrow 可分 116 42.