Lecture I. Nonlinear blowup and planar dynamic system.

In this lecture I'd like to introduce some important notions and tools which is important for studying the smooth implosion of compressible flow.

1) The nonlinear blowup

The idea of blowup origin from the theory of ODE. We consider the following system:

$$\begin{cases} \partial_t u = u^l, \\ u(0) = u_0. \end{cases}$$

First, the classical picard theory yields the local existence and unique of solution $u : [0, T] \to \mathbb{R}$ for some $T < \infty$. Now we concern about the maximal cauchy development of such solution, i.e. find the maximal T^* such that u can be extended to time interval $[0, T^*)$. The equation can be solved by separation of variables, then we will find it heavily depends on the nonlinearity $l = 1 \pm \varepsilon$.

Case I: (Sub)linear, $l \leq 1$, i.e. $l = 1 - \varepsilon$ for some $\varepsilon \geq 0$.

$$\varepsilon = 0, \quad \frac{du}{u} = dt \Rightarrow u = u_0 e^t,$$

$$\varepsilon < 0, \quad \frac{du}{u^{1-\varepsilon}} = dt \Rightarrow u = C(t + \hat{T})^{\frac{1}{\varepsilon}}, \quad C = \sqrt[\varepsilon]{\varepsilon}, \quad \hat{T} = \frac{1}{\varepsilon} u_0^{\varepsilon}.$$

This implies u admit a global existence for low regularity.

Case II: suplinear, $l > 1$, i.e. $l = 1 + \varepsilon$ for some $\varepsilon > 0$,

$$\frac{du}{u^{1+\varepsilon}} = dt \Rightarrow u = \frac{1}{C(T-t)^{\frac{1}{\varepsilon}}}, \quad C = \sqrt[\varepsilon]{\varepsilon}, \quad T^* = \frac{1}{\varepsilon u_0^{\varepsilon}}.$$

This implies the solution blows up in the finite time $T^* = \frac{1}{\varepsilon u_0^{\varepsilon}}$.

(A particular example is $\varepsilon = u_0 = 1$, when $\begin{cases} \partial_t u = u^2 \\ u(0) = 1 \end{cases} \Rightarrow u = \frac{1}{1-t}$, this example is important since many physic equation possess exact 2-order nolinearity, such as transport, NS).

The above discuss give a intuition for singularity: the suplinearity attempt to increase it while sublinear attempt to ease it, which propose the idea of linearization of differential equation.

2) blow up in PDE.

As the modern theory implies, the evolution PDEs can be viewed as a dynamic system which evolute in a Banach space (compared to ODE, which evolute in $\mathbb{R}^d$). Since $\mathbb{R}^d$ have only Euclidean topology, then the only possible blow up for ODE is:

$$|u(t)| \to \infty, \quad \text{as } t \to \infty.$$

However, the blow up in PDE contains much richer pheonomau. For example, if u is a solution of PDE evolute in smooth function space $C^\infty(\mathbb{R}^d)$, then two possible mechanism may occur such that the solution runs out of $C^\infty(\mathbb{R}^d)$:

① $\|u(t)\|_{L^\infty} \to \infty$. This case is quite similar with ODE, which we call a ODE-type blow up or implosion (in fluid mechanic).

② $\|u(t)\|_{L^\infty} \le C$, but $\|\nabla u(t)\|_{L^\infty} \to \infty$. A typical example is the Burgers equation

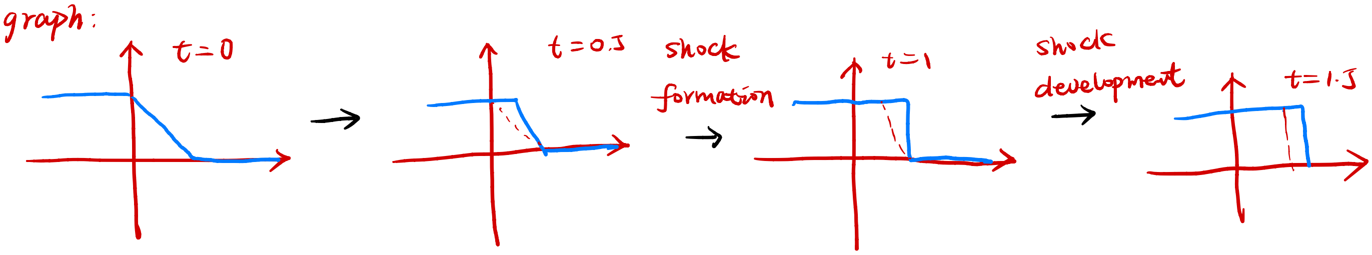
$$u_t + u \cdot \nabla u = 0$$

equipped with initial data

$$u_0(t,x) = \begin{cases} 1, & t<0, \\ 1-t, & t \in [0,1] \\ 0, & t>1. \end{cases}$$

The process when the singularity forms is figured as the right side ($\|\nabla u\|_{L^\infty} \to \infty$).

(graph:



Consequently, this case is usually called a Burgers-type blow up or shock.

Other kind of blowup mechanism may also happens, but we like to discuss in later topic.

2) linearization of differentiation equation.

Now we consider a more general form of ODE in $\mathbb{R}^d$:

(the high order system, such as $\partial_t^2 u = F(u)$ (wave operator). can also be rewritten by set $v = \partial_t u$).

$$\partial_t u = F(u), \quad (\text{no initial data posed}).$$

The idea of linearization is to decompose the system into linear part and nonlinear part, so that we can treat them separately. Support v is a root of F, then it is also a equilibrium solution of $\partial_t u = F(u)$. Now we consider

$$u(t) = v + w(t).$$

then the perturbation $w(t)$ satisfies the following equation:

$$\partial_t w = F(v+w) = Aw + N(w)$$

where $A = F'(v)$, $N(w) = F(v+w) - F(v) = O(|w|)$ by the Taylor expansion of F at v and the fact $F(v) = 0$. Using the Duhamel formula, we can rewrite the perturbation w as the following implicit integration expression:

$$w(t) = \underbrace{e^{tA} w(0)}_{\text{linear part}} + \underbrace{\int_0^t e^{(t-s)A} N(w)}_{\text{nonlinear part}}.$$

(Sometimes we call the solution of the above equation as "mild solution".).

Consequently, we can analyse the properties (particularly singularity and stability in our seminar) of w as well as u.

The idea goes similar for a PDE, when A is a differential operator (such as Laplacian Δ). a typical example is the following nonlinear heat equation:

$$\partial_t u = \Delta u + |u|^{p-1} u.$$

As we notice, the linear part has smooth effect (as $e^{t\Delta}: L^2 \to C^\infty$) where the nonlinear part increase the singularity. The balance of two part finally depend the global existence or finite time blow up of the solution.

dissipation

The idea of linearization is quite significant in the further discussion, which treat the system near some point locally linearly.

3) stability theory: (物理世界的扰动, 混沌蝴蝶)

we initiate the notion of stability by the following ODE system:

$$\partial_t u = F(u,t), \qquad \text{(we say it autonomous if } F \text{ is free with } t\text{).}$$

Suppose $u = u(t)$ is a global solution of above system uniquely generated by data u_0.

Then we say the solution is

① stable if $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $\forall v(t)$ generated by v_0, $|v_0 - u_0| < \eta \Rightarrow |v(t) - u(t)| < \varepsilon$.

② asymptotically stable if stable and $\exists u_0 \in D \subseteq \mathbb{R}^d$, s.t. $\forall v_0 \in D$, $\lim_{t\to\infty} |u(t) - v(t)| = 0$.

Remark: ① The stability concern about the long time behavior compared to the continuity.

(open set $[t_0, \infty)$ vs compact set $[a,b]$).

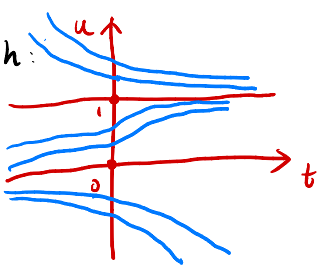
② we alway analyse the stability of equilibrium solutions, as we will find that the stability of general solutions behavier similarly as the near equilibrium points.

we'd like to give some explicit example to illustrate the idea.

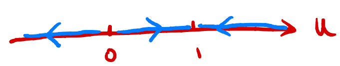
Example:

① $u_t = u(1-u)$, equilibrium point: $u=1$ and $u=0$;

graph:



phase portrait:



(吸引域为 $(0,\infty)$)

$\Rightarrow$ $u=1$ is asymptotically stable;

$\Rightarrow$ $u=0$ is not stable.

② $u_t = A(t)u$, where $A(t)$ is a matrix, equilibrium point $u=0$.

claim: $u = X(t)\cdot C$, then $u=0$ is $\begin{cases} \text{stable iff } X(t) \text{ is bounded.} \\ \text{asymtotically stable iff } |X(t)| \to 0. \end{cases}$

fundamental matrix

(for $A(t) = A$, we have $X(t) = e^{tA}$).

Suppose A formulate as Jordan blocks $A = J_1 \oplus J_2 \oplus \cdots \oplus J_k$, $J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{r_i \times r_i}$.

then the i-th block of fundautal matrix: $e^{tJ_k} = \begin{pmatrix} e^{\lambda_k t} & t e^{\lambda_k t} & \cdots & t^{r_k - 1} e^{\lambda_k t} \\ & e^{\lambda_k t} & \cdots & t^{r_k - 2} e^{\lambda_k t} \\ & & \ddots & \vdots \\ & & & e^{\lambda_k t} \end{pmatrix}$

(competition between exponential $e^{t\lambda}$ and ploynomial growth t^{r_k})

(real part)

$X(t) = e^{tA}$ is bounded if $\begin{cases} \text{all eigenvalue are negative,} \longrightarrow \text{asymtotically for all direction. } V_k \text{.} \\ \text{all eigenvalue are non-positive, and the Jordan block of those with} \\ \text{zero-real part is 1-th order.} \longrightarrow \text{stable for those direction} \end{cases}$

⑤ $u_t = Au + N(u)$, $N(u) = O(|u|)$; $N(0) = 0$.

then case I, III are preserved under perturbation $N(u)$, but case II may no so.

Moreover, we consider the stability in PDE: Consider the nonlinear heat equation:

$$\partial_t u = \Delta u + |u|^{p-1} u.$$

Then we need to study the spectral properties of the Laplace operator Δ. Some significant difficulties and difference follows:

1) The spectrum of differential operator depends heavily on the domain. For example:

Δ on whole space $\mathbb{R}^d$: $\sigma(\Delta) = \sigma_{ess}(\Delta) = (-\infty, 0]$.

Δ on bounded space (compact self-adjoint), $\lambda_i \to 0$. discrete.

$H_0^1 \hookrightarrow\hookrightarrow L^2$

2) No spectral gap: for ODE, since $\{\lambda_i\}$ are finite, if negative, then $\exists$ gap $w > 0$, s.t.

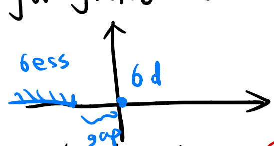
$$\mathrm{Re}\{\lambda_i\} \leq -w, \Rightarrow \text{exponential decay}.$$

$$\Rightarrow |u(t)| \leq e^{-wt} |u(0)|.$$

However, it may holds for PDE, for example, $\sigma(A) = \{-\frac{1}{n}\}_{n=1}^{\infty}$.

another example: reaction-diffusion equation: $u_t = u_{xx} + u - u^3$, $x \in \mathbb{R}$.

Linearized operator for front solution:



3) the operator is not self-adjoint. Stokes operator: $P(-\Delta)$.

pseudo-spectrum: $\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \|(z-A)^{-1}\| > \varepsilon^{-1}\}$.

$$\|(\lambda - A)^{-1}\| = \frac{1}{dist(\lambda - \sigma(A))} \begin{cases} \|(\lambda-A)^{-1}\| \geq \frac{1}{dist}. & \checkmark \\ \|(\lambda-A)^{-1}\| \leq \frac{1}{dist} & \times. \end{cases}$$

$$\sigma(A) = \bigcap \{\lambda \in \mathbb{C} \mid dist(\lambda, \sigma(A)) < \varepsilon\} = \bigcap \underbrace{\{\lambda \in \mathbb{C} \mid \|(\lambda-A)^{-1}\| > \tfrac{1}{\varepsilon}\}}_{\sigma_\varepsilon(A)}.$$

4) phase portrait for planar system

Phase portrait is a important tool to analyse the autonomous dynamic system, which satisfies

① translation invariance: $u(t)$ is a solution $\Rightarrow$ $u(t+c)$ is also a solution;

② no intersection for trajectory: $\Longleftarrow$ by uniqueness + translation invariance.

③ group properties: for fixed u_0, $u_t(u_0) =: u(t, u_0)$, then $\begin{cases} u_0 = id; \\ u_t \circ u_s = u_{t+s}. \end{cases}$

Remark: two autonomous ODE may refer to the same phase portrait. $\frac{d}{dt}u = F(u)$ and $\frac{d}{dt}u = \frac{F(u)}{\sqrt{1+|F(u)|^2}}$.

(相图会损失信息(投影), 但是对稳定性分析很有效).

Example 1: $\begin{cases} \frac{d}{dt}u = -v + u(u^2+v^2-1), \\ \frac{d}{dt}v = u + v(u^2+v^2-1). \end{cases}$

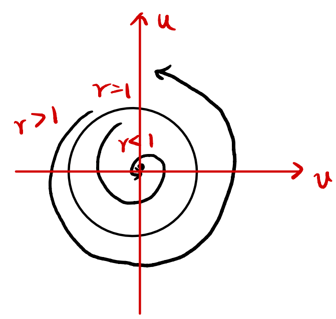
We set Lyapunov funtion: $V = u^2+v^2$ (物理意义: 能量势函数).

which satisfies: $\begin{cases} V(0) = 0, \\ \frac{dV}{dt} = (u^2+v^2)(u^2+v^2-1) \begin{cases} >0, & \text{for } u^2+v^2>1, \\ =0, & \text{for } u^2+v^2=1 \\ <0, & \text{for } u^2+v^2<1. \end{cases} \end{cases}$

in fact, we can solve it use polar coordinate:

$$r = \frac{1}{\sqrt{1-C_1e^{2t}}}, \quad \theta = t+\theta_0, \quad C_1 = (r_0^2-1)/r_0^2.$$

$\longrightarrow$



So we can say equilibrium solution $(u, v) = 0$ is a asymptotic stable.

within precessing domain $r = \sqrt{u^2+v^2} < 1$.

(when the explicit solution is hard to give, we can also use the idea of 1-order approximation to judge the type of equilibrium, after we figure it out for following linear ODE:)

Example 2 $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}$. A is a nondegenerated 2x2 matrix. (or take $P^{-1}AP = J$)

W.L.O.G we assume A is Jordan canonical, i.e. A has following form:

$$① \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad ② \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Case 1: