Lecture 2. Smooth implosion of the compressible flow I.

In this lecture, we'd like to give a brief introduction to the work by Merle - Raphael - Rodnianski - Szeftel, which give a construction of smooth implosion formation for the 3D Navier-Stokes equation under particular settings. Before we dive into the discussion of details, it is proper to introduce the history and related works about this topic.

As the Millennium problem point out, we concern about the development of the flow emerged from (smooth) regular data:

$$\text{regularity} \xleftrightarrow{\text{硬币的正反面}} \text{singularity}$$

will it always exists in the long run or blow up in the finite time?

important works from regularity side:

① Matsumura - Nishida (1980): no vaccum + small perturbation + 3D global existence.

② Huang - Li: vaccum $\begin{cases} \text{small data} + 3D \text{ global existence} \\ \text{Large data} + 2D \text{ (kazhikov) global existence.} \end{cases}$

important works from singularity side: as we emphasize before, the singular pheonomon of PDE is much richer than the ODE because of the diversful topology. Particularly, for the blowup for C^∞ data of compressible flow, there are two case we care most:

1) shock-type singularity: $\|u\|_{L^\infty} \le C$, $\|\nabla u(t)\|_{L^\infty} \xrightarrow{t \to T} \infty$.

   ① shock formation: christodoulou (07) → Luk-Speck (18) → Buckmaster - Shkoller - Vicol (19-22)
   2D → 3D, isentropic → full, zero vorticity → non trivial vorticity

   ② Shock development: ~.

2) implosion-type singularity: $\|u(t)\|_{L^\infty} \to \infty$ as $t \to T$.

   ① Guderley (1948) (non-smooth) imploding shock wave Euler.

   ② Xin (1998). Compacted supported data
   Rozonova (2006). rapidly decaying data
   Yan (2013) isolated mass group data.
   } data with strongly vanishing property, and no insights for the singular nature.

   ③ MRRS (2019) smooth implosion from data with (weak decay).

   ④ Buckmaster (2022) data with constant density
   (using Rieman invariant from shock theory).

Sketch of the idea.

The method to construct smooth implosion is quite different from the former works by Xin, which provide a contradiction argument. but MSSR gives precise depiction for the implosion formation. And they goes under a routine developed for nonlinear singularity (which fields they mainly works on). Though complicate, such method can be decompose into following three part:

① find the kernel singular structure: ODE analysis. (+ dynamic, semiclassical analysis)

Euler

establish the stability of this structure (under perturbation).

NLS ← ② → linear stability: spectral analysis.

NS ← ③ → nonlinear stability: energy method (bootstrap, Brouwer argument).

Remark:

1. Open problems deserves to consider:
   1) the result is not perfect: not all $r>1$ are considered (some of them are denegerated). particularly $d=3$, $r=\frac{5}{3}$ (respond to H_2, N_2 etc.).
      Can we remove the varnishing restriction? (Done by Buckmaster)
   2) can we apply the method to other models? (such as FNS, 2D Kazhikhov model, MHD).
   3) Vaccum problem: The initial data is not allowed vaccum, will it occurs in the long run?
      And if vaccum is allowed, will the singularity happens?
      (this may require a combination of the classical (Huang-Li) and new (MRRS) method, indeed, the regularity structure (such as effective viscous flow) is not considered in this paper, can we obtain better results if it is considered?).

2. How does the schrodinger equation link to the compressible flow:

   via the Madelung transform: $u(t,x)=\sqrt{\rho}e^{i\phi}$, ← stream function.

   let $v=\nabla\Phi$, then
   $$\begin{cases} \partial_t\rho+\nabla\cdot(\rho v)=0, \\ \partial_t v+v\cdot\nabla v+\nabla(k\rho^{\frac{p-1}{2}})=\nabla\left(\frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\right). \end{cases}$$
   ← hydrodynamic flow.

   Link between classical mechanic ⟷ quantum mechanic

   Semiclassical limit: $\hbar\to 0$ $\hbar$: Planck constant (smallest energy unit)

   (类似屏幕上的像素点，像素点越小，画面越逼真，而像素点趋近于0的过程，就是半经典极限).

Illustration of the method: Singular structure.

Roughly saying, the kernel singular structure exhibit in its Euler regime, where the viscous part is treated as a low-order perturbation for such singular structure.

(this belongs to the argument of stability (linear or nonlinear) of this singular structure).

What is the singular structure of Euler regime?

A smooth spherically and radially self-similar blowup solution for the Euler flow (developed from particular set of smooth data). This is exactly the work of the first paper, which mainly use the idea of renormalization, local linearization, the planar phase portrait and semiclassical analysis. (Furthermore, with the auxiliary of numerical method).

Self-similarity: scaling invariance and critical index.

We consider the following nonlinear PDE:

defocusing heat equation: $u_t = \Delta u - |u|^{p-1}u$.

We set scaling $u_\lambda = \frac{1}{\lambda^{\alpha}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$, $\alpha = \frac{2}{p-1}$, such that: if u is a solution of NLH, so does u_λ.

A nautral energy balance is given by $E(u)(t) = \underbrace{\frac{1}{2}\int |\nabla u|^2}_{\text{kinetic}} + \underbrace{\frac{1}{p+1}\int |u|^{p+1}}_{\text{potential}} = E(u_0)$.

Consequence, we seek for a critical index such that

小尺度到大尺度

$$\|\nabla^{s_c} u_\lambda\|_{L^2}^2 = \|\nabla^{s_c} u\|_{L^2}^2 \quad \text{(so that } u_\lambda \text{ does not blow up as } \lambda \to 0).$$

$$s_c = \frac{d}{2} - \frac{2}{p-1}.$$ (depends on the dimension and nonlinearity).

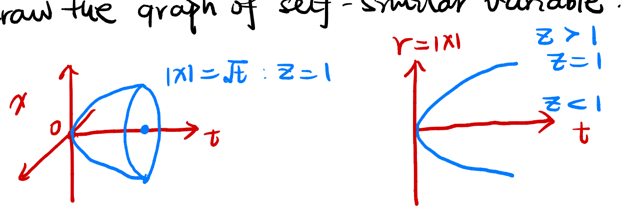
And the global existence happens if we have $s_c < 1$. (which implies $\|\nabla^{s_c} u_\lambda\|_{L^2}^2$ does not blow up).

We call this system is subcritical / critical / supercritical if $s_c < 1$ / $s_c = 1$ / $s_c > 1$.

Moreover, we say u is a self-similar solution if $u = u_\lambda$ for any (t, x). In this case, the solution is totally determined by its profile at the unit time:

$$u(t,x) = \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \overset{\text{set } \lambda(t) = \sqrt{t}}{=} \frac{1}{t^{\frac{1}{p-1}}} u\left(1, \frac{x}{\sqrt{t}}\right) =: \frac{1}{t^{\frac{1}{p-1}}} \underbrace{\hat{u}}_{\text{profile}}\left(\frac{x}{\sqrt{t}}\right), \qquad \hat{u}(z) =: u(1, z).$$

We can draw the graph of self-similar variable:



→ light cone.